

# ON AN ORDER BASED CONSTRUCTION OF A GROUPOID FROM AN INVERSE SEMIGROUP

DANIEL H. LENZ

Fakultät für Mathematik, TU Chemnitz, D - 09107 Chemnitz, Germany

E-mail: dlenz@mathematik.tu-chemnitz.de

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**ABSTRACT.** We present a construction, which assigns two groupoids,  $G_u(\Gamma)$  and  $G_m(\Gamma)$ , to an inverse semigroup  $\Gamma$ . By definition,  $G_m(\Gamma)$  is a subgroupoid (even a reduction) of  $G_u(\Gamma)$ .

The construction unifies known constructions for groupoids. More precisely, the groupoid  $G_u(\Gamma)$  is shown to be isomorphic to the universal groupoid of  $\Gamma$  introduced by Paterson. For  $\Gamma$  arising from graphs resp. tilings, the groupoid  $G_m(\Gamma)$  is the graph groupoid introduced by Kumjian et al. resp. the tiling groupoid introduced by Kellendonk.

We obtain a characterisation of open invariant sets in  $G_m(\Gamma)^{(0)}$  in terms of certain order ideals of  $\Gamma^{(0)}$  for a large class of  $\Gamma$  (including those arising from graphs and from tilings). If  $G_m(\Gamma)$  is essentially principal this gives a characterization of the ideal structure of  $C_{\text{red}}^*(G_m(\Gamma))$  by a theory of Renault. In particular, we then obtain necessary and sufficient conditions on  $\Gamma$  for simplicity of  $C_{\text{red}}^*(G_m(\Gamma))$ .

Our approach relies on a detailed analysis of the order structure of  $\Gamma$ .

## 1. INTRODUCTION

This article is concerned with the construction of groupoids from inverse semigroups and applications to graphs and tilings.

The motivation for our study comes from two sources. The first source is work of Paterson [15, 16] and of Kellendonk [7, 8] (cf. [9, 10] as well). Both Paterson and Kellendonk present constructions assigning groupoids to inverse semigroups. The relationship between their constructions is not quite apparent and it is our aim to present a unified view. More precisely, we will show that the set  $\mathcal{O}(\Gamma)$  of directed subsets of  $\Gamma$  modulo an obvious equivalence relation is an inverse semigroup in the natural way (cf. [8] for related results as well). Restricting the multiplication on  $\mathcal{O}(\Gamma)$  gives then a groupoid  $G_u(\Gamma)$  which is shown to be isomorphic to the universal groupoid of Paterson. It turns out that there is a subset of  $\mathcal{O}(\Gamma)$ , where the restriction of multiplication does not alter the multiplicable pairs. This is the set of minimal elements of  $\mathcal{O}(\Gamma)$ . Thus, this set is a subgroupoid and even a reduction of  $G_u(\Gamma)$ . It will be called  $G_m(\Gamma)$ . Under a rather mild assumption on  $\Gamma$  it can be shown to agree with the groupoid  $H_m(\Gamma)$  introduced by Kellendonk in [8].

The second starting point for this work is given by recent investigations on graphs and their associated groupoids by Kumjian et al. [12]. Kumjian et al. study (among other topics) the ideal structure of  $C^*$ -algebras associated to graphs (cf. [2] and [6] as well for different approaches to these algebras). An important step in their analysis is a characterization of the open invariant sets of the associated groupoid.

Here, it is our aim to present an inverse semigroup based approach to these topics.

More generally, we present an abstract characterization of open invariant sets of  $G_m(\Gamma)^{(0)}$  in terms of certain order ideals of  $\Gamma^{(0)}$  for quite general  $\Gamma$ .

The connection to graphs is then made by showing that  $G_m(\Gamma)$  is actually isomorphic to the graph groupoid of Kumjian et al. if  $\Gamma$  is a suitable graph inverse semigroup. Restricting the theory developed for general  $\Gamma$  to the graph case, we then recover the mentioned results of Kumjian et al..

Let us now discuss these points in some more detail. As discussed by Renault [19], every topological groupoid  $G$  gives rise to an inverse semigroup viz its ample semigroup  $G^a$ . If the groupoid is ample, this ample semigroup determines the topology of the groupoid. This naturally raises the question whether every semigroup can be faithfully represented as subsemigroup of  $G^a$  determining the topology of  $G$  for a suitable groupoid  $G$ .

This question has been thoroughly addressed by Paterson [15, 16]. It turns out that such a representation is not unique in general and may not even exist if  $G$  is required to be Hausdorff [15]. On the other hand, Paterson presents a construction assigning what he calls the universal groupoid  $G_u(\Gamma)$  to an inverse semigroup  $\Gamma$ . This is an ample but not necessarily Hausdorff groupoid on whose ample semigroup  $\Gamma$  can be faithfully represented. It determines all other so called  $\Gamma$ -groupoids and also determines the representation theory of  $\Gamma$  [16]. Paterson also shows that  $G_u(\Gamma)$  is Hausdorff if  $\Gamma$  is  $E$ -unitary.

This approach relies on a generalization of Kumjians theory of localization [11] developed by Paterson in [15, 16]. Here, a localization means a suitable action of an inverse semigroup  $\Gamma$  on a space  $X$ . As shown by Paterson this gives rise to an  $r$ -discrete groupoid  $G(X, \Gamma)$  whose  $C^*$ -algebra is isomorphic to  $C_0(X) \rtimes \Gamma$ , where the partial crossed product is taken the sense of Sieben [21]. In fact, results of Paterson [16] and Sieben [21, 22] (cf. [18] as well) show, roughly speaking, that there is a one-to-one correspondence between  $r$ -discrete groupoids and (partial) actions of inverse semigroups on suitable spaces (at least on the level of  $C^*$ -algebras). Related topics have also been studied by Exel in [4] and Nica in [14].

While  $G_u(\Gamma)$  has very nice universal features its relation to groupoids arising in concrete examples (as tiling groupoids or graph groupoids) is not quite clear.

On the other hand there is a different construction due to Kellendonk [8] assigning a groupoid  $H_m(\Gamma)$  to an inverse semigroup. In the context of tilings this construction gives the tiling groupoid. Moreover, as we show below this construction also gives the graph groupoid when applied to a suitable inverse semigroup associated to a graph. The construction is based on suitable maximal ordered sequences in  $\Gamma$ .

Therefore, it is our first aim here to investigate the relation between the groupoid introduced by Kellendonk and that introduced by Paterson. To do so we will have another and somewhat more systematic look at the order based considerations done by Kellendonk in [8]. This will give a unified construction to both groupoids

and clarify their relationship. This approach also leads to interesting additional information on their respective topologies.

Our construction is simple and does not use the theory of localizations or any actions of  $\Gamma$  on a space. It can rather be seen as a kind of order completion of  $\Gamma$  (cf. [3] for related material in general lattices). Moreover, it gives immediately a lattice type condition on  $\Gamma$ , which we term (L), for  $G_u(\Gamma)$  to be Hausdorff (Corollary 4.10). Condition (L) just means suitable existence of minima and is strictly weaker than  $E$ -unitarity and  $0$ - $E$ -unitarity. If  $\Gamma$  satisfies (L), the topology of  $G_u(\Gamma)$  is not only Hausdorff but even admits a different description. Namely, it can then simply be described as the topology inherited from the product topology of  $\{0, 1\}^\Gamma$  under the natural injection (Lemma 4.9).

If  $\Gamma$  satisfies (L), then the topology of  $G_m(\Gamma)$  also has very nice features and a particularly simple basis (Lemma 6.7). This can be used to show that in this case the groupoid  $H_m(\Gamma)$  introduced by Kellendonk is isomorphic to  $G_m(\Gamma)$ . In fact,  $H_m(\Gamma)$  and  $G_m(\Gamma)$  always agree as sets but the topology might be different. These results clarify the relationship of  $H_m(\Gamma)$  and  $G_u(\Gamma)$ .

As (L) is satisfied for tiling inverse semigroups our construction gives the tiling groupoid in this case. The condition (L) is also satisfied for a suitable inverse semigroup associated to graphs. In this case  $G_m(\Gamma)$  can be shown to be the graph groupoid introduced by Kumjian et al. (Theorem 5). Thus, the construction easily produces two important classes of groupoids.

In the general case  $G_m(\Gamma)$  is a reduction of  $G_u(\Gamma)$  (in the set theoretical sense) on some invariant set  $E$ . We present two types of general conditions on  $\Gamma$  for  $E$  to be closed yielding that  $G_m(\Gamma)$  is a reduction of  $G_u(\Gamma)$  in the topological sense (Proposition 6.4 and Proposition 6.6). Both types of condition are met in the tiling case and in the graph case. These general conditions are important because they imply, in particular, that  $G_m(\Gamma)$  is locally compact which is not clear in general.

Our study of  $G_m(\Gamma)$  leads us to a certain inverse subsemigroup  $\tilde{\Gamma}$  which is a quotient of  $\Gamma$ . This quotient  $\tilde{\Gamma}$  gives a precise version of how  $\Gamma$  can be considered as an inverse subsemigroup of the inverse semigroup of  $G_m(\Gamma)$ -sets in  $G_m(\Gamma)$ .

We then investigate the lattice of open invariant sets in  $G_m(\Gamma)^{(0)}$ . This lattice plays a key role in the ideal theory of the  $C^*$ -algebra  $C_{\text{red}}^*(G_m(\Gamma))$  due to a theory of Renault [19, 20]. Our main result there gives a characterization of this lattice in terms of a lattice of certain order ideals in  $\Gamma^{(0)}$  (Lemma 7.7). This result can be used to provide necessary and sufficient conditions on  $\Gamma$  for non-existence of non-trivial open invariant subsets of  $G_m(\Gamma)^{(0)}$  (Lemma 7.8). If  $G_m(\Gamma)$  is essentially principal, these Lemmata completely describe the ideal theory of  $C_{\text{red}}^*(G_m(\Gamma))$  and give a necessary and sufficient condition on  $\Gamma$  for simplicity of  $C_{\text{red}}^*(G_m(\Gamma))$  (Theorem 4).

Unfortunately, we have not been able to find convincing conditions on  $\Gamma$  for  $G_m(\Gamma)$  to be essentially principal. For  $\Gamma$  arising from (suitable) graphs,  $G_m(\Gamma)$  is essentially principal by arguments of [12] and we recover the results of [12]. For  $\Gamma$  arising from tilings there is a simple condition for principality of the corresponding groupoids. Thus, we can find a description of ideals of  $C_{\text{red}}^*(G_m(\Gamma))$  in this case as well.

This paper is organized as follows: In Section 2, we review several known facts on (the order of) inverse semigroups. In particular, we show that an inverse semigroup

gives rise to two groupoids, one arising by restricting the multiplication the other consisting of minimal elements.

Section 3 contains the basic constructions showing that the set of directed sets in an inverse semigroup is again an inverse semigroup. In Section 4 we study certain aspects of  $G_u(\Gamma)$  in some detail. This concerns in particular the topology. We prove Lemma 4.9 and Corollary 4.10. Section 5 is devoted to a study of the inverse semigroup  $\tilde{\Gamma}$ . Section 6 presents a general study of  $G_m(\Gamma)$ . It contains Lemma 6.7, and Theorem 3.

Section 8 is devoted to applications to graphs. We prove Theorem 5 and show how the material of the preceeding sections can be used to recover some results of Kumjian et al.. Finally, in Section 9, we recall results of [7, 8] on tilings and provide a study of ideal theory of the algebras  $C_{\text{red}}^*(G_m(\Gamma))$  in this case. This underlines the similarity between the tiling case and the graph case.

## 2. PRELIMINARIES

In this section we fix some notation and review basic facts from inverse semigroup theory. In particular, we collect properties concerning the order structure of an inverse semigroup. For the general theory of inverse semigroups we refer the reader to e.g. [13, 16]. There, one can also find those proofs which are omitted below. The results of this section will be used tacitly in the sequel.

Let  $\Gamma$  be an inverse semigroup. This means that  $\Gamma$  is a semigroup and to each  $x \in \Gamma$  there exists a unique  $i(x) \in \Gamma$  with  $x i(x) x = x$  and  $i(x) x i(x) = i(x)$ . The element  $i(x)$  is denoted by  $x^{-1}$  and called the inverse of  $x$ . The map  $x \mapsto x^{-1}$  is an involution. By  $\Gamma^{(0)}$  we denote the units of  $\Gamma$ , i.e. the set of  $p$  with  $p = pp^{-1}$ . The units are just the idempotents and commute. On  $\Gamma$  we have the relation  $\prec$ , where  $x \prec y$ , whenever  $xy^{-1} = xx^{-1}$ . If  $x \prec y$ , then  $x$  is said to be a predecessor of  $y$  and  $y$  is said to be a successor of  $x$ . Alternatively,  $x$  is said to be smaller than  $y$ . The following proposition is well known and easy to prove.

**Proposition 2.1.** *For  $x, y \in \Gamma$  the following are equivalent:*

- (i)  $x \prec y$ . (ii)  $x^{-1} \prec y^{-1}$ . (iii)  $\exists p \in \Gamma^{(0)} x = py$ . (iv)  $\exists q \in \Gamma^{(0)} x = yq$ .

Using this proposition, it is not hard to see that  $\prec$  is an order on  $\Gamma$ , i.e. a reflexive, transitive relation s.t.  $x \prec y$  and  $y \prec x$  implies  $x = y$ . Moreover, we easily deduce from the proposition the following proposition.

**Proposition 2.2.** *If  $x_1 \prec y_1$  and  $x_2 \prec y_2$ , then  $x_1 x_2 \prec y_1 y_2$ .*

As  $\prec$  is an order, there can not be more than one  $z \in \Gamma$  with  $z \prec x$  for every  $x \in \Gamma$ . A simple calculation shows that such a  $z$  satisfies  $zx = z = xz$  for every  $x \in \Gamma$ . Therefore, it will be denoted by 0.  $\Gamma$  is said to be an inverse semigroup with zero if it contains such a  $z$ . In the sequel we will sometimes write conditions of the form  $0 \neq z \in \Gamma$ . This is meant to mean that  $z$  is not zero if  $\Gamma$  has a zero and to be a vacuous condition if  $\Gamma$  does not contain a zero.

An element  $x \in \Gamma$  is called minimal if it is not zero and  $y \prec x$  and  $y \neq 0$  implies  $y = x$ . The set of minimal elements in  $\Gamma$  is denoted by  $\Gamma_{\min}$ . The following proposition follows easily from the above two propositions.

**Proposition 2.3.** *For  $x \in \Gamma$  the following are equivalent:*

- (i)  $x$  is minimal. (ii)  $x^{-1}$  is minimal. (iii)  $x^{-1}x$  is minimal. (iv)  $xx^{-1}$  is minimal.

Moreover, we have the following result.

**Proposition 2.4.** *For  $x, y \in \Gamma_{\min}$  the following are equivalent:*

(i)  $xy \neq 0$ . (ii)  $x^{-1}x = yy^{-1}$ . (iii)  $xy \in \Gamma_{\min}$ .

*Proof.* (i)  $\implies$  (ii). By  $0 \neq xy = xyy^{-1}y$ , we have  $xyy^{-1} \neq 0$ . By minimality of  $x$ , this gives  $x = xy^{-1}y$ , which in turn implies  $x^{-1}x = x^{-1}xyy^{-1}$ . Similarly, we calculate  $yy^{-1} = x^{-1}xyy^{-1}$  and (ii) follows.

(ii)  $\implies$  (iii). By (ii), we have  $x = xx^{-1}x = xyy^{-1}$  and  $xy \neq 0$  follows. Moreover, by minimality of  $y$  we have  $xyp = xy$  for every  $p \in \Gamma^{(0)}$  with  $xyp \neq 0$ .

(iii)  $\implies$  (i). This is immediate from the definition of minimality.  $\square$ .

The order  $\prec$  will in general not be a semilattice. However, one can still ask for the existence of a largest common predecessor of  $x$  and  $y$  given that there exist common predecessor. If such a largest common predecessor exists it must be unique and will be denoted by  $x \wedge y$ . The existence of such largest predecessors will be of crucial importance in our dealing with topological properties of the groupoids in question. We include a short discussion.

**Definition 2.5.** *An inverse semigroup is said to satisfy the lattice condition (L), if for any  $x, y \in \Gamma$ , with a common predecessor not equal to zero there exists a largest common predecessor.*

The following definition gives a well known criterion for the existence of largest common predecessors. Recall that an order ideal  $\mathcal{I}$  in  $\Gamma$  is a set with  $\{y : y \prec x\} \subset \mathcal{I}$  for every  $x \in \mathcal{I}$ . By an inverse order ideal, we mean a set  $\mathcal{I}$  in  $\Gamma$  with  $\{y : x \prec y\} \subset \mathcal{I}$  for every  $x \in \mathcal{I}$ .

**Definition 2.6.** *An inverse semigroup called  $E$ -unitary if  $\Gamma^{(0)}$  is an inverse order ideal. An inverse semigroup is called 0- $E$ -unitary if  $\Gamma^{(0)} \setminus \{0\}$  is an inverse order ideal.*

**Remark 1.** (a) Apparently, we have  $E$ -unitary  $\implies$  0- $E$ -unitary. But, of course,  $E$ -unitary is essentially used for inverse semigroups without zero, while 0- $E$ -unitarity is the right notion for inverse semigroups with zero.

(b) The condition (L) is strictly weaker than  $E$ -unitarity and 0- $E$ -unitarity. This can be seen by considering a groupoid  $G$  and inverse subsemigroups of the inverse semigroup  $S(G)$  of its  $G$ -sets

**Proposition 2.7.** *Let  $\Gamma$  be a  $E$ -unitary (or 0- $E$ -unitary). If  $x, y \in \Gamma$  have a common predecessor  $z$  not equal to zero, then there exists a largest such  $z$ . It is given by  $yx^{-1}y = xy^{-1}x$ .*

While  $E$ -unitarity resp. 0- $E$ -unitarity has been used, when studying topological properties of groupoids associated to inverse semigroups [8, 16], it turns out that our considerations need only the weaker condition (L).

Actually, the existence of  $x \wedge y$  can be shown for arbitrary  $\Gamma$  under certain conditions on  $x, y$ . This is investigated in the next proposition.

**Proposition 2.8.** *Let  $x, y \in \Gamma$  with a common successor  $z \in \Gamma$  be given. If there exists a common predecessor of  $x$  and  $y$  not equal to zero then there exists a largest common predecessor. It is given by  $xx^{-1}yy^{-1}x = xx^{-1}yy^{-1}y$ .*

**Proof.** As  $x$  and  $y$  are smaller than  $z$  the elements  $p = xx^{-1}, q = yy^{-1}$  belong to  $\Gamma^{(0)}$  with  $x = pz$  and  $y = qz$ . As  $x$  and  $y$  have common predecessors not equal to

zero, the product  $pqz$  is not zero. It is obviously smaller than  $x$  and  $y$ . Moreover, it is not hard to show that any  $r$  smaller than  $x$  and  $y$  must be smaller than  $pqz$ .  $\square$

Let us now turn to groupoids. Recall that a groupoid is a set  $G$  together with a partially defined associative multiplication  $*$  and an involution  $x \mapsto x^{-1}$  satisfying the following conditions [19]:

- (G1)  $(x^{-1})^{-1} = x$ .
- (G2) If  $x * y$  and  $y * z$  exist, then  $x * y * z$  exists as well.
- (G3)  $x^{-1} * x$  exists and if  $x * y$  exists as well then  $x^{-1} * x * y = y$ .
- (G4)  $x * x^{-1}$  exists and if  $z * x$  exists as well then  $z * x * x^{-1} = z$ .

Now, there is an immediate way to construct two groupoids from  $\Gamma$ . We start with the following result contained in e.g. Proposition 1.0.1 of [16].

**Proposition 2.9.** *Let  $\Gamma$  be an inverse semigroup. Then,  $\Gamma$  with its usual inversion and multiplication defined by  $x * y = xy$  if and only if  $x^{-1}x = yy^{-1}$  is a groupoid.*

**Definition 2.10.** *Let  $G(\Gamma)$  be the groupoid associated to  $\Gamma$  in the foregoing proposition.*

**Proposition 2.11.** *The set of minimal elements of  $\Gamma$  with involution from  $\Gamma$  and multiplication defined whenever the product is not zero, is a subgroupoid of  $G(\Gamma)$ .*

*Proof.* By Proposition 2.3 the set of minimal elements is closed under inversion. By Proposition 2.4, it is further closed under the multiplication in  $G(\Gamma)$ .  $\square$

**Definition 2.12.** *Let  $M(\Gamma)$  be the subgroupoid of  $G(\Gamma)$  consisting of minimal elements of  $\Gamma$ .*

To state our next result, we recall some more facts. A subset  $E$  of the units  $G^{(0)}$  of a groupoid  $G$  is called invariant if for  $e \in E$  and  $g \in G$  with  $e = g^{-1}g$  the element  $gg^{-1}$  belong to  $E$  as well. If  $E$  is invariant, the (set theoretic) reduction  $G_E$  of  $G$  to  $E$  is the subgroupoid of  $G$  consisting of all elements  $g \in G$  with  $g^{-1}g \in E$  (which by invariance implies  $gg^{-1} \in E$  as well). In the context of topological groupoids, the invariant set  $E$  is further required to be closed in  $G^{(0)}$ . We will then speak of the topological reduction  $G_E$ .

**Proposition 2.13.** *With  $E \equiv \Gamma^{(0)} \cap \Gamma_{\min} \equiv \Gamma_{\min}^{(0)}$  the equality  $M(\Gamma) = G(\Gamma)_E$  holds.*

*Proof.* The set  $E$  is invariant by Proposition 2.3. Thus,  $G(\Gamma)_E$  is indeed a groupoid. As the groupoid structure is induced by the groupoid structure of  $G(\Gamma)$  for both  $M(\Gamma)$  and  $G(\Gamma)_E$ , it is enough to show that the underlying sets are equal. This is easy.  $\square$

In a groupoid  $G$ , the set of its  $G$ -sets is an inverse semigroup. This inverse semigroup will be denoted by  $S(G)$ .

### 3. THE BASIC CONSTRUCTION

In this section we will show that the set of downward directed subsets of  $\Gamma$  modulo a certain equivalence relation is an inverse semigroup with respect to the obvious multiplicative structure. Lemma 3.2 is strongly related to results of [7]. This is discussed at the end of Section 6.

**Definition 3.1.** A subset  $A$  in  $\Gamma$  is called (downward) directed if for any  $x, y \in A$ , there exists a  $z \in A$  with  $z \prec x, y$ . The set of all directed subsets of  $\Gamma$  is denoted by  $\mathcal{F}(\Gamma)$ .

On  $\mathcal{F}(\Gamma)$ , we define the relation  $\prec$  by  $A \prec B$ , if for any  $b \in B$ , there exists an  $a \in A$  with  $a \prec b$ . Moreover, we define  $AB$  by  $AB \equiv \{ab : a \in A, b \in B\}$  and  $A^{-1}$  by  $A^{-1} = \{a^{-1} : a \in A\}$ . The corresponding sets are indeed directed by the results the first section. Moreover, we set  $A \sim B$ , whenever  $A \prec B$  and  $B \prec A$ . It is not hard to see that  $\sim$  is an equivalence relation on  $\mathcal{F}(\Gamma)$ . We set  $\mathcal{O}(\Gamma) = \mathcal{F}(\Gamma) / \sim$ . Representatives of  $X, Y \in \mathcal{O}(\Gamma)$  will be denoted by  $\dot{X}$  and  $\dot{Y}$ . The class of  $A \in \mathcal{F}(\Gamma)$  will be denoted by  $[A]$ . On  $\mathcal{O}(\Gamma)$ , we define a multiplication by

$$XY \equiv [\dot{X}\dot{Y}],$$

where  $\dot{X}$  and  $\dot{Y}$  are arbitrary representatives of  $X$  and  $Y$ . It is easy to check that this is a well defined associative multiplication. Moreover, we define a map  $i : \mathcal{O}(\Gamma) \rightarrow \mathcal{O}(\Gamma)$  by  $i(X) \equiv [\dot{X}^{-1}]$ , where again  $\dot{X}$  is a representative of  $X$  and this is well defined. The following lemma holds.

**Lemma 3.2.** The set  $\mathcal{O}(\Gamma)$  with multiplication and inversion  $X^{-1} \equiv i(X)$  is an inverse semigroup. The relation  $X \prec Y$  holds for  $X, Y \in \mathcal{O}(\Gamma)$  if  $\dot{X} \prec \dot{Y}$  holds for some (all) representatives  $\dot{X}$  of  $X$  and  $\dot{Y}$  of  $Y$ .

*Proof.* We first show that each  $X \in \mathcal{O}(\Gamma)$  has a unique inverse given by  $i(X)$ . Existence follows easily from

$$\dot{X} = \{xx^{-1}x : x \in \dot{X}\} \sim \{x_1x_2^{-1}x_3 : x_1, x_2, x_3 \in \dot{X}\} = \dot{X}\dot{X}^{-1}\dot{X}.$$

To show uniqueness, let  $X$  and  $Y$  be given with representatives  $\dot{X}$  and  $\dot{Y}$  and assume (1)  $XYX = X$  and (2)  $YXY = Y$ . By (1), we have

$$\dot{X}^{-1} \sim \dot{X}^{-1}\dot{X}\dot{X}^{-1} \sim \dot{X}^{-1}\dot{X}\dot{Y}\dot{X}\dot{X}^{-1} \prec \dot{Y}$$

yielding  $i(X) \prec Y$ . Similarly, by (2), we arrive at  $Y \prec i(X)$ . Putting this together, we obtain the desired uniqueness result. This shows that  $\mathcal{O}(\Gamma)$  is indeed an inverse semigroup. Using this, it is not hard to obtain the statement about the order.  $\square$

Now, we combine this construction with the results of the first section on groupoids associated to inverse semigroups.

**Definition 3.3.** (a) The groupoid  $G_u(\Gamma) \equiv G(\mathcal{O}(\Gamma))$  is called the universal groupoid associated to  $\Gamma$ .

(b) The groupoid  $G_m(\Gamma) \equiv M(\mathcal{O}(\Gamma))$  is called the minimal groupoid of  $\Gamma$ .

From the considerations of the first section, in particular, Proposition 2.13, we immediately infer the following proposition.

**Proposition 3.4.**  $G_m(\Gamma) = G_u(\Gamma)_{\mathcal{O}(\Gamma)_{\min}^{(0)}}$ .

Let us also note the following simple fact.

**Proposition 3.5.** Let  $\Gamma$  be an inverse semigroup with zero. Then  $[B] \neq 0$  holds for every directed set  $B$  with  $0 \notin B$ .

4. THE GROUPOID  $G_u(\Gamma)$ 

In this section we introduce a topology on  $G_u(\Gamma)$  making it into a topological  $r$ -discrete groupoid. This topology has a basis consisting of compact  $G_u(\Gamma)$ -sets. We also show, that  $G_u(\Gamma)$  with this topology is actually isomorphic to the universal groupoid introduced by Paterson in [15, 16].

In the sequel we simply write  $x$  instead of  $[\{x\}] \in \mathcal{O}(\Gamma)$  for  $x \in \Gamma$ . In particular, we write  $X \prec x$  instead of  $X \prec [\{x\}]$  for  $X \in \mathcal{O}(\Gamma)$ . Note that we have  $X = xX^{-1}X = XX^{-1}x$  for  $X \prec x$ . This will be used several times in the sequel. For  $x \in \Gamma$ , we set  $U_x \equiv \{X \in G_u(\Gamma) : X \prec x\}$ . For  $x, x_1, \dots, x_n \in \Gamma$  with  $x_1, \dots, x_n \prec x$ , we set

$$U_{x;x_1,\dots,x_n} \equiv U_x \cap U_{x_1}^c \cap \dots \cap U_{x_n}^c.$$

Here,  $U_x^c$  is the complement of  $U_x$  in  $\mathcal{O}(\Gamma)$ . We will show that the family of these  $U_{x;x_1,\dots,x_n}$  gives a basis of a topology. To do so, we need the following proposition.

**Proposition 4.1.** *For  $X \in G_u(\Gamma)$  and  $x_1, \dots, x_n \prec x$  and  $y_1, \dots, y_m \prec y$  in  $\Gamma$  with  $X \in U_{x;x_1,\dots,x_n} \cap U_{y;y_1,\dots,y_m}$ , there exist  $z_1, \dots, z_k \prec z$  with  $z \prec x, y$  and  $X \in U_{z;z_1,\dots,z_k} \subset U_{x;x_1,\dots,x_n} \cap U_{y;y_1,\dots,y_m}$ .*

*Proof.* Let  $p_j$  and  $q_l$  in  $\Gamma^{(0)}$  be given with  $x_j = xp_j$  and  $y_l = yq_l$ ,  $j = 1, \dots, n$ ,  $l = 1, \dots, m$ . By  $X \in U_x \cap U_y$ , there exists  $z \in \Gamma$  with  $X \prec z \prec x, y$ . Thus, there exist  $p, q \in \Gamma^{(0)}$  with  $z = xp = yq = xpq = ypq$ . Of course, it suffices to show

$$X \in U_{z;zp_1,\dots,zp_n,zq_1,\dots,zq_m} \subset U_{x;x_1,\dots,x_n} \cap U_{y;y_1,\dots,y_m}.$$

It is straightforward to show that  $X$  belongs to  $U_{z;zp_1,\dots,zp_n,zq_1,\dots,zq_m}$ . So, let us now show that  $Y \in U_{z;zp_1,\dots,zp_n,zq_1,\dots,zq_m}$  belongs to  $U_{x;x_1,\dots,x_n} \cap U_{y;y_1,\dots,y_m}$  as well. By  $Y \prec z$  we have  $Y \prec x$  and  $Y \prec y$ . Thus, it remains to show that  $Y$  does neither belong to  $U_{x_j}$  nor to  $U_{y_l}$  for arbitrary  $j$  and  $l$  as above. Assume  $Y \prec xp_j$ . By  $Y \prec z$ , this gives the contradiction

$$Y = YY^{-1}Y \prec xp_j z^{-1}xp_j = xp_j z^{-1}zp_j = xz^{-1}zp_j = zp_j,$$

where we used  $z \prec x$  twice. Similarly, we show that  $Y \prec yq_l$  cannot hold. The proposition follows.  $\square$

The proposition implies that the family of all sets in  $G_u(\Gamma)$  which are a union of sets of the form  $U_{x;x_1,\dots,x_n}$  is a topology.

**Definition 4.2.** *The topology  $\mathcal{T}$  on  $G_u(\Gamma)$  is the family of sets which are unions of sets of the form  $U_{x;x_1,\dots,x_n}$ .*

**Proposition 4.3.** *Inversion and multiplication in  $G_u(\Gamma)$  are continuous with respect to  $\mathcal{T}$ .*

*Proof.* The statement about inversion is obvious. To show that multiplication is continuous, let  $Z = X * Y \in U_{z;z_1,\dots,z_n}$  be given. Let  $p_j, q_j \in \Gamma^{(0)}$  be given with  $z_j = p_j z = z q_j$  for  $j = 1, \dots, n$ . There exist  $x, y \in \Gamma$  with  $X \prec x$ ,  $Y \prec y$  and  $xy \prec z$ . As  $X * Y$  exists in  $G_u(\Gamma)$ , we have  $X^{-1}X = YY^{-1}$  and we can assume w.l.o.g.  $x^{-1}x = yy^{-1}$ . Now, it is straightforward to show that  $X \in U_{x;p_1x,\dots,p_nx}$  and  $Y \in U_{y,yq_1,\dots,yq_n}$ . Thus, it remains to show that for  $A \in U_{x;p_1x,\dots,p_nx}$  and  $B \in U_{y,yq_1,\dots,yq_n}$  the product  $A * B$  belongs to  $U_{z;z_1,\dots,z_n}$  (if it exists). Apparently,  $A * B$  belongs to  $U_{xy} \subset U_z$ . Assume  $A * B \prec z_j$  for some  $j$ . Then, there exist  $a, b \in \Gamma$  with  $A \prec a$  and  $B \prec b$  and  $ab \prec z_j$ . Again, as  $AB$  exists in  $G_u(\Gamma)$ , we can



assume w.l.o.g.  $a^{-1}a = bb^{-1}$ . Moreover, we can assume w.l.o.g.  $a \prec x$  and  $b \prec y$  as  $A \in U_x$  and  $B \in U_y$ . This gives

$$\begin{aligned} a = aa^{-1}a &\prec ax^{-1}x = aa^{-1}ax^{-1}x = abb^{-1}x^{-1}x \\ &\prec z_j b^{-1}x^{-1}x \prec z_j y^{-1}x^{-1}x \prec p_j z z^{-1}x \prec p_j x. \end{aligned}$$

This gives a contradiction, as  $A$  does not belong to  $U_{p_j x}$ . The proposition follows.  $\square$ .

The proposition says that  $G_u(\Gamma)$  with the topology  $\mathcal{T}$  is a topological groupoid. Let us now further investigate the topology. Even though the topology need not be Hausdorff, it has certain separation properties. The following proposition shows in particular, that the topology is  $T_1$ . Thus, a converging net cannot have more than one limit.

**Proposition 4.4.** (a) For arbitrary  $X \neq Y \in G_u(\Gamma)$ , there exists  $z \prec x \in \Gamma$  with  $X \in U_{x;z}$  and  $Y \notin U_{x;z}$ .  
(b) The set  $G_u(\Gamma)^{(0)}$  is closed in  $G_u(\Gamma)$ .

*Proof.* (a) Consider first the case  $Y \prec X$  (and  $X \neq Y$ ). Let  $x \in \Gamma$  with  $X \prec x$  be given. Then, there exists an  $y \prec x$  with  $Y \prec y$  and not  $X \prec y$ . This gives  $X \in U_{x;y}$  and  $Y \notin U_{x;y}$ . On the other hand if  $Y \prec X$  does not hold, then there exists an  $x \in \Gamma$  with  $X \prec x$  and not  $Y \prec x$  and we infer  $X \in U_x$  and  $Y \notin U_x$ .  
(b) It suffices to show that, for every converging net  $(P_i)$  in  $G_u(\Gamma)^{(0)}$ , the limit  $P$  belongs to  $G_u(\Gamma)^{(0)}$  i.e. satisfies  $P = PP^{-1}$ . But this is immediate from (a) and continuity of multiplication.  $\square$

**Proposition 4.5.** Let  $p_1, \dots, p_n \prec p \in \Gamma^{(0)}$  be given. Let  $x \in \Gamma$  with  $p \prec x^{-1}x$  be given. Then  $U_{xp;xp_1,\dots,xp_n} = xU_{p;p_1,\dots,p_n}$ .

*Proof.* This follows easily from  $X = xX^{-1}X$  and the fact that  $X^{-1}X \prec q$  if and only if  $xX^{-1}X \prec xq$  for  $q \prec x^{-1}x$  and  $X \prec x$ .  $\square$

Combining the foregoing propositions, we infer the following corollary.

**Corollary 4.6.** The maps  $s : U_{x;x_1,\dots,x_n} \longrightarrow U_{x^{-1}x;x_1^{-1}x_1,\dots,x_n^{-1}x_n}$ ,  $X \mapsto X^{-1}X$  and  $r : U_{x;x_1,\dots,x_n} \longrightarrow U_{xx^{-1};x_1x_1^{-1},\dots,x_nx_n^{-1}}$ ,  $X \mapsto XX^{-1}$  are homeomorphisms.

*Proof.* We only show the statement about  $s$ . The statement about  $r$  follows similarly. By the foregoing proposition, the map  $s^* : U_{x^{-1}x;x_1^{-1}x_1,\dots,x_n^{-1}x_n} \longrightarrow U_{x;x_1,\dots,x_n}$ ,  $P \mapsto xP$  is surjective. By

$$(xP)^{-1}xP = Px^{-1}xP = PP = P,$$

$s^*$  is injective as well. Moreover, we see that  $s$  and  $s^*$  are inverse to each other and  $s$  is therefore a bijection. By Proposition 4.3, the map  $s$  is continuous. Using Proposition 4.1 and Proposition 4.5, one can also infer that  $s^*$  is continuous.  $\square$

Recall that a groupoid is called  $r$ -discrete if its topology has a basis of sets on which  $r$  and  $s$  are homeomorphic. Thus, the foregoing corollary says that  $G_u(\Gamma)$  is  $r$ -discrete.

**Proposition 4.7.** For arbitrary  $x_1, \dots, x_n \prec x \in \Gamma$  the set  $U_{x;x_1,\dots,x_n}$  is compact.

*Proof.* By the foregoing Corollary, it suffices to consider  $U_{p;p_1,\dots,p_n}$  with  $p_1, \dots, p_n \prec p \in \Gamma^{(0)}$ . This, however, is just a reformulation of the well known properties of the maximal ideal space of the commutative Banach algebra  $l^1(\Gamma^{(0)})$  (cf. [16] as well). We include a short sketch for completeness. Apparently, the map  $j : G_u(\Gamma)^{(0)} \longrightarrow \{0, 1\}^{\Gamma^{(0)}}$  with  $j(P)(q) = 1$  if  $P \prec q$  and  $j(P)(q) = 0$  otherwise, is injective (cf. Lemma 4.9 as well). Moreover, if  $\{0, 1\}$  carries the discrete topology and  $\{0, 1\}^{\Gamma^{(0)}}$  is given the product topology, then the topology in  $G_u(\Gamma)^{(0)}$  is easily seen to be the topology induced by this injection. Thus, it remains to show that  $j(G_u(\Gamma)^{(0)})$  is closed in  $\{0, 1\}^{\Gamma^{(0)}}$ . So, assume that the net  $(j(P_i))$  converges to  $f \in \{0, 1\}^{\Gamma^{(0)}}$ . Then, it is not hard to see that  $\{q \in \Gamma^{(0)} : f(q) = 1\}$  is a directed inverse order ideal and  $f = j(P)$  with  $P = [\{q \in \Gamma^{(0)} : f(q) = 1\}]$ .  $\square$

Finally, we have the following proposition concerning the algebraic properties of  $x \mapsto U_x$ .

**Proposition 4.8.** *The map  $U : \Gamma \longrightarrow S(G_u(\Gamma))$ ,  $x \mapsto U_x$  is an injective homomorphism of inverse semigroups.*

*Proof.* By Corollary 4.6 the sets  $U_x$  are indeed  $G_u(\Gamma)$ -sets. Thus,  $V$  maps into  $S(G_u(\Gamma))$ . Apparently,  $V$  preserves the involution. Thus, it only remains to show  $U_{xy} = U_x U_y$ . The inclusion  $\supset$  is obvious. Let now  $Z \in U_{xy}$  be given. By  $Z \prec xy$ , we have  $x^{-1}Z \prec y$  and  $Zy^{-1} \prec x$  as well as  $Zy^{-1}x^{-1} = ZZ^{-1}$ . This implies  $Z = ZZ^{-1}Z = Zy^{-1}x^{-1}Z = XY$  with  $X \equiv Zy^{-1}$  and  $Y \equiv x^{-1}Z$ . It remains to show that  $X$  and  $Y$  are composable in the sense of the groupoid  $G_u(\Gamma)$  i.e. that  $X^{-1}X = YY^{-1}$ . But this follows from

$$X^{-1}X = yZ^{-1}Zy^{-1} = x^{-1}Zy^{-1} = x^{-1}ZZ^{-1}x = YY^{-1},$$

where we used  $Zy^{-1} \prec x$  and  $x^{-1}Z \prec y$ . Injectivity is simple.  $\square$ .

We summarize our considerations in the following theorem.

**Theorem 1.** *The groupoid  $G_u(\Gamma)$  is a topological groupoid with basis of topology given by the family of sets  $U_{x;x_1,\dots,x_n}$  for arbitrary  $x_1, \dots, x_n \prec x \in \Gamma$ . These sets are compact  $G_u(\Gamma)$ -sets on which  $r$  and  $s$  are homeomorphisms. The map  $U : \Gamma \longrightarrow S(G_u(\Gamma))$  is an injective homomorphism of inverse semigroups.*

Let us now consider the Hausdorff properties of  $G_u(\Gamma)$ . By the proof of Proposition 4.7, its unit space is Hausdorff. However, in general  $G_u(\Gamma)$  will not be Hausdorff. We will show that, for  $\Gamma$  satisfying condition (L), there a simple alternative description of the topology of  $G_u(\Gamma)$ . This will then give that  $G_u(\Gamma)$  is Hausdorff if  $\Gamma$  satisfies (L) (cf. Corollary 4.10 below).

Consider the map  $j : G_u(\Gamma) \longrightarrow \{0, 1\}^\Gamma$  with  $j(X)(x) = 1$  if  $X \prec x$  and  $j(X)(x) = 0$  otherwise. Let  $\{0, 1\}$  carry discrete topology and let  $\{0, 1\}^\Gamma$  be given the product topology. We have the following lemma.

**Lemma 4.9.** *The map  $j$  is injective. If  $\Gamma$  satisfies (L), the topology induced on  $G_u(\Gamma)$  from  $\{0, 1\}^\Gamma$  agrees with  $\mathcal{T}$ .*

*Proof.* It is not hard to show  $X = [\{y : X \prec y\}]$ . Thus, if  $X \neq Y$ , then there exists w.l.o.g. an  $x \in \Gamma$  with  $X \prec x$  but not  $Y \prec x$ . This gives  $j(X)(x) = 1$  and  $j(Y)(x) = 0$  and injectivity follows.

To show that the induced topology agrees with  $\mathcal{T}$ , we have to show that for arbitrary  $X \in G_u(\Gamma)$  and  $x_1, \dots, x_n, y_1, \dots, y_m$  with  $X \in U_{x_1} \cap \dots \cap U_{x_n} \cap U_{y_1}^c \cap \dots \cap U_{y_m}^c$ , there exist  $z_1, \dots, z_k \prec z$  with

$$(1) \quad X \in U_{z; z_1, \dots, z_k} \subset U_{x_1} \cap \dots \cap U_{x_n} \cap U_{y_1}^c \cap \dots \cap U_{y_m}^c.$$

By  $X \in U_{x_1} \cap \dots \cap U_{x_n}$ , there exists an  $x \in \Gamma$  with  $X \prec x \prec x_1, \dots, x_n$ . By (L), we can then set  $z \equiv x_1 \wedge \dots \wedge x_n$ . Apparently, we have  $X \in U_z \subset U_{x_1} \cap \dots \cap U_{x_n}$ . Similarly, we can define  $z_j = z \wedge y_j$  for every  $j$  with  $U_z \cap U_{y_j} \neq \emptyset$ . Assume w.l.o.g. that the set of these  $j$  is given by  $\{1, \dots, k\}$ . By construction (1) holds.  $\square$ .

**Corollary 4.10.** *If  $\Gamma$  satisfies (L), then  $G_u(\Gamma)$  is Hausdorff.*

**Remark 2.** In [16] it is shown that  $G_u(\Gamma)$  is Hausdorff if  $\Gamma$  is  $E$ -unitary. As  $E$ -unitary implies (L), the foregoing Corollary gives a strengthening of this result.

We close this section with a discussion of the isomorphy between  $G_u(\Gamma)$  and the universal groupoid,  $H_u(\Gamma)$  constructed by Paterson [15, 16]. His construction proceeds in three steps:

- A certain inverse semigroup  $\Gamma'$  containing  $\Gamma$  is shown to act on the space  $X$  of semicharacters s.t.  $(X, \Gamma')$  is a localization.
- By Patersons extension of Kumjians theory of localizations [11, 16], there exist a groupoid  $G(X, \Gamma')$  for this localization.
- The groupoid  $H_u(\Gamma) = G(X, \Gamma')$  can be expressed in terms of  $X$  and  $\Gamma$  only.

We refrain from discussing the theory of localizations here and just give the description of  $H_u(\Gamma)$  in terms of  $X$  and  $\Gamma$  according to Theorem 4.3.1 of [16]. In our discussion we will identify the space of semicharacters used in [16] with  $\mathcal{O}(\Gamma)^{(0)}$  (cf. proof of Proposition 4.7 above and discussion in Section 4.3 of [16]). Moreover, we will use the notation introduced above. In particular, the action of  $\Gamma$  on  $X$  will be written accordingly. Using these adoptions to our setting, the groupoid  $H_u(\Gamma)$  can be described as follows:

It consists of equivalence classes  $[P, x]$ , of pairs  $(P, x)$  with  $P \in \mathcal{O}(\Gamma)^{(0)}$ ,  $x \in \Gamma$  with  $P \prec xx^{-1}$ . Here, two pairs  $(P, x)$  and  $(\tilde{P}, \tilde{x})$  are identified if  $P = \tilde{P}$  and there exists an  $p \in \Gamma$  with  $P \prec p$  and  $px = p\tilde{x}$ . The involution is given by  $[P, x]^* \equiv [x^{-1}Px, x^{-1}]$  and the multiplication is given by  $[P, x] * [x^{-1}Px, y] \equiv [P, xy]$ . A basis of the topology is given by sets of the form  $\{[P, x] : P \in U_{p; p_1, \dots, p_n}\}$ .

Given these reformulations of the Paterson construction, the proof of the following theorem is a simple exercise.

**Theorem 2.** *The map  $J : G_u(\Gamma) \longrightarrow H_u(\Gamma)$ ,  $J(X) \equiv [X^{-1}X, x]$  with an arbitrary  $x$  with  $X \prec x$  is an isomorphism of topological groupoids with inverse map  $K$  given by  $K : H_u(\Gamma) \longrightarrow G_u(\Gamma)$ ,  $K([P, x]) \equiv Px$ .*

## 5. THE INVERSE SEMIGROUP $\tilde{\Gamma}$

In this section we introduce and investigate a certain quotient of  $\Gamma$ , which we call  $\tilde{\Gamma}$ . The relevance of this quotient will become apparent in the next sections when we deal with  $G_m(\Gamma)$ . It will then turn out that  $\tilde{\Gamma}$  and not  $\Gamma$  is the appropriate semigroup to phrase certain features of  $G_m(\Gamma)$ .

**Definition 5.1.** For  $n \in \mathbb{N}$  and  $x, x_1, \dots, x_n \in \Gamma$ , we set  $x < (x_1, \dots, x_n)$  if for every  $y \prec x$ ,  $y \neq 0$ , there exists  $z \in \Gamma$ ,  $z \neq 0$  and  $j \in \{1, \dots, n\}$  with  $z \prec y, x_j$ . If  $n = 1$ , we write  $x < x_1$  instead of  $x < (x_1)$ .

The relation  $<$  is not an order. However, it can be shown to induce an order on a certain quotient of  $\Gamma$  by a standard procedure in the treatment of preorders. This is investigated next.

The relation  $x <> y$  if and only if  $x < y$  and  $y < x$  can easily be seen to give an equivalence relation on  $\Gamma$ . The quotient  $\tilde{\Gamma}$  is then defined by  $\tilde{\Gamma} \equiv \Gamma / <>$ . Let  $\pi : \Gamma \rightarrow \tilde{\Gamma}$  be the canonical projection.

Let us collect a few useful properties of  $<$ .

**Proposition 5.2.**  $x < y$  implies  $x^{-1} < y^{-1}$  as well as  $xz < yz$  and  $zx < zy$ .

*Proof.* This is straightforward.  $\square$

**Proposition 5.3.** There exists a unique inverse semigroup structure on  $\tilde{\Gamma}$  making  $\pi$  into a homomorphism of inverse semigroups. The relation  $x < y$  holds for  $x, y \in \Gamma$  if and only if  $\pi(x) \prec \pi(y)$ .

*Proof.* The uniqueness statement is obvious. Let us now show existence of the desired semigroup structure. Using the foregoing proposition, we infer that the sets  $\pi(\pi^{-1}(a)^{-1})$  resp.  $\pi(\pi^{-1}(a)\pi^{-1}(b))$  contain exactly one element. Thus, we can define  $a^{-1}$  resp.  $ba$  by  $\pi(\pi^{-1}(a)^{-1})$  resp.  $\pi(\pi^{-1}(a)\pi^{-1}(b))$ . Let us now show that the inverse is unique. Let  $x, y \in \Gamma$  and  $a, b \in \tilde{\Gamma}$  with  $a = \pi(x)$  and  $b = \pi(y)$  and  $aba = a$  and  $bab = b$  be given. By  $aba = a$ , we have  $x < xyx$  implying  $x^{-1} = x^{-1}xx^{-1} < x^{-1}xyxx^{-1} < y$ . Similarly, we infer  $y^{-1} < x$  and  $y^{-1} <> x$  follows. Thus,  $\tilde{\Gamma}$  is indeed an inverse semigroup and  $\pi$  is an homomorphism of inverse semigroups.

It remains to show the statement about the order. Let  $x, y \in \Gamma$  with  $x < y$  be given. We have to show  $\pi(x)\pi(x^{-1}) = \pi(x)\pi(y)^{-1}$  i.e.  $xx^{-1} <> xy^{-1}$ . By  $x < y$  and the foregoing proposition, we infer

$$(2) \quad xx^{-1} < xy^{-1}, \quad xx^{-1} < yx^{-1}.$$

Thus, it remains to show  $xy^{-1} < xx^{-1}$ . But this follows from

$$xy^{-1} = xx^{-1}xx^{-1}xy^{-1} < xx^{-1}yx^{-1}xy^{-1} \prec xx^{-1}.$$

Here, we used (2) to obtain the estimate  $<$ . Conversely, assume  $\pi(x) \prec \pi(y)$ . This gives easily  $x < yx^{-1}x \prec y$ .  $\square$

Next, we study  $\tilde{\Gamma}$  for  $\Gamma$  satisfying (L). Our main tool in this study is the following proposition.

**Proposition 5.4.** Let  $\Gamma$  satisfy (L). If  $x, y, z \in \Gamma$  satisfy  $0 \neq z < x, y$ , then  $x \wedge y \wedge z$  exists and is not equal to zero.

*Proof.* By  $z \prec z$  and  $z < x$ , we derive from (L) that  $0 \neq z \wedge x$  exists. By  $0 \neq z \wedge x \prec z$  and  $z < y$ , we infer, again by (L), that  $0 \neq z \wedge x \wedge y$  exists.  $\square$

We can now deduce two further properties of  $\tilde{\Gamma}$ .

**Proposition 5.5.** (a) If  $\Gamma$  satisfies (L), then  $\tilde{\Gamma}$  satisfies (L) as well.  
 (b) Let  $\Gamma$  be an inverse semigroup with zero satisfying (L). Then  $\tilde{\Gamma} = (\tilde{\Gamma})^{\sim}$

*Proof.* (a) Let  $0 \neq c \prec a, b \in \tilde{\Gamma}$  be given. Choose  $x, y, z \in \Gamma$  with  $\pi(x) = a$ ,  $\pi(y) = b$  and  $\pi(z) = c$ . By Proposition 5.3, we then have  $0 \neq z < x, y$ . By the foregoing proposition,  $0 \neq x \wedge y$  exists. By  $x \wedge y < x, y$  (even  $x \wedge y \prec x, y$ ) and Proposition 5.3, we then have  $\pi(x \wedge y) \prec \pi(x), \pi(y)$ . Moreover, straightforward argument show that  $z < x \wedge y$  holds yielding  $c = \pi(z) \prec \pi(x \wedge y)$ . Combining these estimates, we infer  $\pi(x \wedge y) = \pi(x) \wedge \pi(y)$ .

(b) It suffices to show  $x < y$  whenever  $\pi(x) < \pi(y)$  for  $x, y \in \Gamma$ . So, assume  $\pi(x) < \pi(y)$ . W.l.o.g. we can assume  $0 \neq \pi(x)$ . Let  $0 \neq z \prec x$  be given. Then, we have  $\pi(z) \prec \pi(x)$  and by  $\pi(x) < \pi(y)$ , there exists  $r \in \Gamma$  with  $0 \neq \pi(r) \prec \pi(z), \pi(y), \pi(x)$ . This gives  $0 \neq r < z, y, x$  by Proposition 5.3. By Proposition 5.4, we then infer  $0 \neq z \wedge y \wedge x \wedge r$  and  $x < y$  follows.  $\square$

For later use we also note the following proposition.

**Proposition 5.6.** (a) *The relation  $<$  is transitive, i.e. for  $x < (x_1, \dots, x_k)$  with  $x_j < (x_{j,1}, \dots, x_{j,n(j)})$ ,  $j = 1, \dots, k$ ,  $x_{j,l} \in \Gamma$  suitable, the relation  $x < (x_{1,1}, \dots, x_{1,n(1)}, \dots, x_{k,1}, \dots, x_{k,n(k)})$  holds.*  
(b) *If  $p < (p_1, \dots, p_n)$  and  $p \prec x^{-1}x$  for suitable  $p, p_1, \dots, p_n \in \Gamma^{(0)}$  and  $x \in \Gamma$ , then  $xpx^{-1} < (xp_1x^{-1}, \dots, xp_nx^{-1})$ .*

*Proof.* (a) This is straightforward.

(b) Let  $0 \neq q \prec xpx^{-1}$  be given. By  $xpx^{-1} = xx^{-1}xpx^{-1}xx^{-1}$ , this implies  $q = xx^{-1}qxx^{-1}$  yielding  $x^{-1}qx \neq 0$ . Furthermore, we have  $x^{-1}qx \prec x^{-1}xpx^{-1}x = p$ . Thus, there exist  $r \in \Gamma^{(0)} \setminus \{0\}$  and  $j \in \{1, \dots, n\}$  with  $r \prec x^{-1}qx$  and  $r \prec p_j$ . This implies  $0 \neq rrx^{-1} \prec xx^{-1}qxx^{-1} \prec q$  and  $rrx^{-1} \prec xp_jx^{-1}$  finishing the proof of (b).  $\square$

## 6. THE GROUPOID $G_m(\Gamma)$

By the general theory presented in Section 2 the groupoid  $G_m(\Gamma)$  is a subgroupoid of  $G_u(\Gamma)$  and in fact a reduction in the set theoretical sense. Thus, it inherits the topology from  $G_u(\Gamma)$  and is a topological  $r$ -discrete groupoid. A basis of the topology is given by the sets

$$V_{x;x_1, \dots, x_n} \equiv U_{x;x_1, \dots, x_n} \cap G_m(\Gamma)$$

for arbitrary  $x_1, \dots, x_n \prec x \in \Gamma$ . However, it is not clear in general, whether the sets  $V_{x;x_1, \dots, x_n}$  are compact. Moreover, it is not clear whether  $V_x$  is actually not empty. So, we start by discussing conditions on non-emptiness and compactness of  $V_x$ ,  $x \in \Gamma$ .

**Proposition 6.1.** *If  $\Gamma$  contains a zero, then there exists for every  $Y \in \mathcal{O}(\Gamma)$ ,  $Y \neq 0$ , an  $X \in \mathcal{O}(\Gamma)_{\min}$  with  $X \prec Y$ . In particular,  $V_x \neq \emptyset$  for every  $x \neq 0$ .*

*Proof.* Let  $\dot{Y}$  be a representative of  $Y$ . By  $Y \neq 0$ , we have  $0 \notin \dot{Y}$ . Consider the family of directed sets containing  $\dot{Y}$  but not containing 0. The usual inclusion gives a partial order on this family. Application of Zorns Lemma, then gives a maximal element  $B$  in this family. This element does not contain zero and as  $\Gamma$  contains a zero, we see  $[B] \neq 0$ . By construction  $[B]$  is minimal and precedes  $Y$ .  $\square$

**Proposition 6.2.** (a) *Let  $\Gamma$  be an inverse semigroup with zero satisfying (L). Then, the following are equivalent:*

(i)  $x < (x_1, \dots, x_n)$ . (ii)  $V_x \subset V_{x_1} \cup \dots \cup V_{x_n}$ .

In particular,  $V_x = V_y$  if and only if  $x < y$  and  $y < x$ .

(b) For arbitrary  $\Gamma$  with zero (not necessarily satisfying (L)), the equivalence of (i) and (ii) holds, whenever  $x, x_1, \dots, x_n$  belong all to  $\Gamma^{(0)}$ .

*Proof.* (a) (i)  $\implies$  (ii). Let  $X \in V_x$  be given. Then,  $A \equiv \{y : y \prec x, X \prec y\}$  is a representative of  $X$ . Set  $A_j \equiv \{y \wedge x_j : y \in A \text{ s.t. } 0 \neq y \wedge x_j \text{ exists}\}$ . By (i) and (L), there exists a  $j$  with  $A_j \prec A$ . This gives  $[A_j] \prec X$ . As  $\Gamma$  has a zero, we have  $[A_j] \neq 0$  and by minimality of  $X$ , we infer  $X = [A_j]$ . As  $[A_j]$  belongs to  $V_{x_j}$ , the statement (ii) is proven.

(ii)  $\implies$  (i). Let  $y \prec x$ ,  $y \neq 0$ , be given. As  $\Gamma$  has a zero, there exists by Proposition 6.1 an  $Y \in \mathcal{O}(\Gamma)_{\min}$  with  $Y \prec y$ . This implies  $Y \in V_y \subset V_x$ . By (ii), we infer  $Y \in V_{x_j}$  i.e.  $Y \prec x_j$  for a suitable  $j$ . Thus,  $y$  and  $x_j$  have a common predecessor not equal to zero.

(b) This follows easily from the fact that existence of largest predecessors is always valid on  $\Gamma^{(0)}$ .  $\square$

We can now study compactness properties of the  $V_x$ ,  $x \in \Gamma$ .

**Proposition 6.3.** *The following are equivalent:*

- (i) For arbitrary  $x_1, \dots, x_n \prec x \in \Gamma$  the set  $V_{x; x_1, \dots, x_n}$  is compact.
- (ii) The set  $G_m(\Gamma)^{(0)}$  is closed in  $G_u(\Gamma)$ .
- (iii)  $G_m(\Gamma)$  is a topological reduction of  $G_u(\Gamma)$ .

*Proof.* The equivalence of (ii) and (iii) is immediate from Proposition 2.13. The implication (ii)  $\implies$  (i) is immediate from Proposition 4.5 and Proposition 4.7. Thus, it remains to show (i)  $\implies$  (ii). Let  $(P_i)$  be a net in  $G_m(\Gamma)^{(0)}$  converging in  $G_u(\Gamma)$  to  $P \in G_u(\Gamma)$ . By Proposition 4.4 (b),  $P$  belongs to  $G_u(\Gamma)^{(0)}$ . Thus,  $P$  belongs to  $U_p$  for a suitable  $p \in \Gamma^{(0)}$ . Then  $P_i$  belongs to  $V_p$  for large  $i$ . As  $V_p$  is compact,  $P = \lim P_i$  must then belong to  $V_p \subset G_m(\Gamma)^{(0)}$  as well.  $\square$

Of course, it might be difficult to decide whether one of the conditions of the proposition holds. Thus, let us now give a criterion, which despite its simplicity can be checked for certain concrete semigroups e.g those arising in the context of tilings and graphs.

A function  $R : \Gamma \longrightarrow I$  with  $I = [0, \infty)$  or  $I = [0, \infty]$  is called a radiusfunction if it satisfies (R1)  $R(x^{-1}) = R(x)$ , (R2)  $R(xy) \geq \min\{R(x), R(y)\}$ , (R3)  $R(y) \leq R(x)$  for  $x \prec y$ . A radiusfunction  $R$  on  $\Gamma$ , gives rise to a radiusfunction on  $\mathcal{O}(\Gamma)$ , called  $R$  again by  $R(X) \equiv \sup\{R(x) : X \prec x\}$ . A radiusfunction is called admissible if  $R(X) = \infty$  if and only if  $X \in \mathcal{O}(\Gamma)_{\min}$ .

**Remark 3.** The definition of radiusfunction just says that  $R$  is a dual prehomomorphism from  $\Gamma$  into  $(I, \wedge)$ . Here, we set  $xy \equiv x \wedge y \equiv \min\{x, y\}$  for  $x, y \in I$ .

**Proposition 6.4.** *If  $R$  is an admissible and continuous radiusfunction on  $G_u(\Gamma)$ , then  $V_{x; x_1, \dots, x_n}$  is compact for arbitrary  $x_1, \dots, x_n \prec x \in \Gamma$ .*

*Proof.* By the foregoing proposition, it suffices to show that  $G_m(\Gamma)^{(0)}$  is closed in  $G_u(\Gamma)^{(0)}$ . But this is immediate from continuity of  $R$  and  $G_m(\Gamma)^{(0)} = G_u(\Gamma)^{(0)} \cap \{X \in \mathcal{O}(\Gamma) : R(X) = \infty\}$ .  $\square$

**Remark 4.** (a) It is not hard to show that any radiusfunction must be lower semi-continuous.

(b) The radiusfunctions arising in the context of graphs or tilings are admissible and have the additional property that for  $x \in \Gamma$  the maximum  $\pi_r(x) \equiv \max\{y : x \prec y, R(y) \geq r\}$  exists for arbitrary  $r \leq R(x)$ . In this case it is possible to show that  $\pi_r(X) \equiv \pi_r(x)$  for arbitrary  $x \in \Gamma, X \in \mathcal{O}(\Gamma)$  with  $X \prec x$  and  $R(x) \geq r$  is well defined and satisfies (1)  $\pi_r(X) = \pi_r(Y)$  for  $X \prec Y$  and (2)  $\pi_s(X) = \pi_s(\pi_r(X))$  for  $s \leq r$  and  $R(X) \geq r$ . Thus, in this case one can find a canonical representative  $(\pi_n(X))_{n \in \mathbb{N}}$  of  $X \in \mathcal{O}(\Gamma)_{\min}$ . This can be used to show that  $G_m(\Gamma)$  can be considered as a kind of metric completion of  $\mathcal{O}(\Gamma)$  w.r.t.  $d(X, Y) \equiv \exp(-\sup\{r \geq 0 : \pi_r(X) = \pi_r(Y)\})$ . In the context of tilings this has been investigated in [8]

Let us give another condition for closedness of  $G_m(\Gamma)^{(0)}$  in  $G_u(\Gamma)^{(0)}$ . This condition is local in the sense that it can be checked by only considering  $\Gamma$  (and not  $\mathcal{O}(\Gamma)$ ).

**Definition 6.5.** *The inverse semigroup  $\Gamma$  is said to satisfy the trapping condition (T), if  $\Gamma$  contains a zero and for every  $p, q \in \Gamma^{(0)}$  with  $q \prec p$  there exist  $p_1, \dots, p_n \in \Gamma^{(0)}$  with  $p_j \prec p$ ,  $j = 1, \dots, n$ , and*

- $p < (p_1, \dots, p_n, q)$ .
- For every  $j \in \{1, \dots, n\}$  either  $p_j \prec q$  or  $p_j q = 0$ .

**Proposition 6.6.** *Let  $\Gamma$  satisfy (T). Then  $G_m(\Gamma)^{(0)}$  is closed in  $G_u(\Gamma)^{(0)}$ .*

*Proof.* Let  $(P_i)$  be a net in  $G_m(\Gamma)^{(0)}$  converging in  $G_u(\Gamma)$  to  $P$ . As  $G_u(\Gamma)^{(0)}$  is closed in  $G_u(\Gamma)$ , the element  $P$  belongs to  $G_u(\Gamma)^{(0)}$ . Next, we show that  $P$  is not zero. Assume the contrary. As  $\Gamma$  contains a zero, this implies  $P \in U_0$  yielding the contradiction  $0 \neq P_i \in U_0$  for large  $i$ .

So, it suffices to show that every  $Q \neq 0$ ,  $Q \prec P$  agrees with  $P$ . Let such a  $Q$  be given and assume  $P \neq Q$ . Then, there exist  $p, q \in \Gamma^{(0)}$  with  $q \prec p$  and  $Q \prec q$ ,  $P \prec p$  but not  $P \prec q$ . Choose  $p_1, \dots, p_n$  according to (T) for  $q \prec p$ . Then, we have  $V_p \subset V_{p_1} \cup V_{p_n} \cup V_q$  by (T) and Proposition 6.2 (b). Then, it is not hard to see that there exists a subnet  $(P_k)$  of  $(P_i)$  converging to  $P$  as well and  $(P_k) \subset V_{p_j}$  for a suitable  $j$ . By  $P_k \in V_{p_j} \subset U_{p_j}$  and compactness of  $U_{p_j}$ , we infer  $P \in U_{p_j}$  i.e.  $P \prec p_j$ . There are two cases:

- Case 1.*  $p_j \prec q$ : In this case we arrive at the contradiction  $P \prec p_j \prec q$ .  
*Case 2.*  $p_j q = 0$ : In this case we have  $Pq = Pp_j q = P0 = 0$  contradicting  $0 \neq Q = Qq \prec Pq$ .  $\square$

If  $\Gamma$  satisfies (L) the topology of  $G_m(\Gamma)$  has a particularly nice basis.

**Lemma 6.7.** *If  $\Gamma$  satisfies (L) and has a zero, then the family of sets  $V_x$ ,  $x \in \Gamma$ , is a basis of the topology of  $G_m(\Gamma)$ .*

*Proof.* It suffices to show that for arbitrary  $X \in G_m(\Gamma)$  and  $z_1, \dots, z_n \prec z \in \Gamma$  with  $X \in V_{z; z_1, \dots, z_n}$ , we have  $X \in V_x \subset V_{z; z_1, \dots, z_n}$  for a suitable  $x$ . Assume the contrary. Thus, there exists  $X \in G_u(\Gamma)$  s.t. for every  $x$  with  $X \prec x$  the set  $V_x \cap (V_{z_1} \cup \dots \cup V_{z_n})$  is not empty. We therefore must have  $X = [A_j]$  with a suitable  $j$  for  $A_j \equiv \{x : X \prec x, V_x \cap V_{z_j} \neq \emptyset\}$ . Assume w.l.o.g.  $j = 1$ . By (L), then the minimum  $x \wedge z_1$  exists for arbitrary  $X \prec x$  and is not zero. Moreover, the construction gives  $[\{x \wedge z_1 : X \prec x\}] \prec X$ . and  $[\{x \wedge z_1 : X \prec x\}]$  is not zero, as  $\Gamma$  has a zero. By minimality of  $X$ , this gives  $X = [\{x \wedge z_1 : x \in \dot{X}\}]$  and the contradiction  $X \prec z_1$  follows.  $\square$

Let us now study the map  $V : \Gamma \longrightarrow S(G_m(\Gamma))$ ,  $x \mapsto V_x$ .

**Proposition 6.8.** *The map  $V : \Gamma \longrightarrow S(G_m(\Gamma))$ ,  $x \mapsto V_x$  is an homomorphism of inverse semigroups. If  $\Gamma$  satisfies (L) and contains a zero,  $V(\Gamma)$  is canonically isomorphic to  $\tilde{\Gamma}$  by  $V_x \mapsto \pi(x)$ .*

*Proof.* The first statement can be shown with the same proof as Proposition 4.8. The second statement then follows from Proposition 6.2.  $\square$

**Remark 5.** *The proposition shows, in particular, that the map  $V$  on  $\Gamma$  is, unlike  $U$ , not necessary injective. Nevertheless, it is still possible to show that  $G_m(\Gamma)$  is isomorphic to  $G_m(V(\Gamma))$ , whenever  $\Gamma$  satisfies (L).*

In this context, we also have the following result. Recall that the ample semigroup of a groupoid  $G$  is the inverse semigroup consisting of all compact open  $G$ -sets. A groupoid is called ample if this semigroup is a basis of the topology. The result shows that the construction of  $G_m(\Gamma)$  does not yield anything new if  $\Gamma$  is already (large part of) an ample semigroup of an ample groupoid.

**Theorem 3.** *Let  $\Gamma$  be a subsemigroup of the inverse semigroup of an ample Hausdorff groupoid  $G$ . Assume that  $\Gamma$  is closed under intersections (which implies (L)) and that  $\Gamma$  is a basis of the topology of  $G$ . Then  $G_m(\Gamma) \simeq G$ .*

*Proof.* This is the analogue in our setting to a result of [8]. Thus, we only briefly sketch the idea. To each point  $g \in G$  we associate the set  $A_g$  consisting of all  $x \in \Gamma$  with  $g \in x$ . This set is directed i.e. belongs to  $\mathcal{O}(\Gamma)$ , as  $\Gamma$  is closed under intersections. Using that  $G$  is Hausdorff, one easily sees that  $[A_g]$  must be minimal i.e. belong to  $G_m(\Gamma)$ . Conversely, using finite intersection property of compact sets, it is not hard to see that  $g(X) \equiv \cap_{X \prec x} x$  is not empty for every  $X \in G_m(\Gamma)$ . By minimality of  $X$  and again as  $G$  is Hausdorff, the set  $g(X)$  must then consist of only one point, which is denoted by  $g(X)$ . The maps  $g \mapsto [A_g]$  and  $X \mapsto g(X)$  are groupoid homomorphisms and inverse to each other.  $\square$

The considerations of this section suggest to distinguish the class of inverse semigroups with zero which satisfy (L) and give rise to an ample groupoid  $G_m(\Gamma)$ . Thus, we introduce the following definition.

**Definition 6.9.** *The inverse semigroup  $\Gamma$  is said to satisfy condition (LC), if it contains a zero, satisfies (L) and  $G_m(\Gamma)^{(0)}$  is closed in  $G_u(\Gamma)^{(0)}$ .*

**Remark 6.** We will see that the inverse semigroup arising from the graphs considered in [12] and those arising from the tilings in [8] satisfy (LC).

Let us close this section with a comparison to the corresponding results of [8]. In [8] almost-groupoids are considered. An almost-groupoid is essentially an inverse semigroup with zero, whose zero has been removed and whose multiplication has been restricted accordingly. Thus, the inverse semigroups underlying the considerations in [8] all have a zero. The set of totally ordered sequences modulo the obvious equivalence relation is shown in [8] to be an almost-groupoid whose set of minimal elements is a groupoid and even a topological groupoid if equipped with the topology generated by the  $V_x$ ,  $x \in \Gamma$  (in our notation). Call it  $H_m(\Gamma)$ .

The considerations of this section extend the corresponding considerations of [7] in some ways.



First of all, the relationship between  $H_m(\Gamma)$  and  $G_u(\Gamma)$  is made explicit. More precisely, we show that  $G_m(\Gamma)$  is a subgroupoid and even a set theoretical reduction of  $G_u(\Gamma)$ . Now, it can easily be seen that the groupoid  $H_m(\Gamma)$  agrees with  $G_m(\Gamma)$  as a set, but the topology might be different. Here, Lemma 6.7 is important. It shows that  $G_m(\Gamma)$  and  $H_m(\Gamma)$  agree as topological groupoids if  $\Gamma$  satisfies (L).

Second of all we study the question whether  $G_m(\Gamma)$  has a basis of compact open sets. In [8] this question is only addressed in the tiling case. Here, we show that the existence of such a basis is essentially equivalent to  $G_m(\Gamma)$  being a topological reduction of  $G_u(\Gamma)$ . Moreover, we give two simple sufficient criteria on  $\Gamma$  for this being the case. Both criteria are met in both the tiling and graph case.

Finally, as a minor point, let us remark that our treatment is slightly more flexible as we use directed sets rather than totally ordered sequences.

## 7. OPEN INVARIANT SUBSETS OF $G_m(\Gamma)^{(0)}$

It is well known [19] that in arbitrary locally compact groupoids  $G$ , each open invariant subset  $U$  of  $G^{(0)}$  gives rise to an ideal in  $C_{\text{red}}^*(G)$  which is canonically isomorphic to  $C_{\text{red}}^*(G_U)$ . If  $G$  is an essentially principal groupoid (s. below for definition), then every ideal in  $C_{\text{red}}^*(G)$  arises in this way. Thus, the investigation of open invariant subsets of  $G_m(\Gamma)^{(0)}$  is of primary importance.

In this section we relate the open invariant subsets of  $G_m(\Gamma)^{(0)}$  to certain order ideals in  $\Gamma^{(0)}$  (Lemma 7.7). If  $G_m(\Gamma)$  is essentially principal, this gives a complete characterization of the ideals in  $C_{\text{red}}^*(G_m(\Gamma))$  (Theorem 4). For  $\Gamma$  satisfying (LC), we also give a necessary and sufficient condition on  $\Gamma$  for  $G_m(\Gamma)^{(0)}$  not to admit nontrivial invariant open sets (Lemma 7.8). This gives in particular a necessary and sufficient condition for simplicity of  $C_{\text{red}}^*(G_m(\Gamma))$  whenever  $G_m(\Gamma)$  is essentially principal. The semigroups we have in mind are those satisfying (LC), even though some results of this section are actually valid for more general inverse semigroups.

We start with a discussion of invariance. Let  $X \in G_m(\Gamma)$  be given,. Let  $x \in \Gamma$  with  $X \prec x$  be given. Then, we have  $X = XX^{-1}X = Px = xQ$  with  $P = XX^{-1}, Q = X^{-1}X$  in  $G_m(\Gamma)^{(0)}$ . This shows  $X^{-1}X = x^{-1}Px$  and  $XX^{-1} = xQx^{-1}$ . These considerations easily imply the following proposition.

**Proposition 7.1.** (a) For a subset  $E$  of  $G_m(\Gamma)^{(0)}$  the following are equivalent:

(i)  $E$  is invariant.

(ii) For every  $P \in E$  and  $x \in \Gamma$  with  $P \prec xx^{-1}$ , the element  $x^{-1}Px$  belongs to  $E$ .

(b)  $G_m(\Gamma)_P^P \equiv \{X : XX^{-1} = X^{-1}X = P\} = \{P\}$  if and only if every  $x \in \Gamma$  with  $x^{-1}Px = P$  and  $P \prec xx^{-1}$  satisfies  $P \prec x$ .

**Definition 7.2.** An element  $P \in G_m(\Gamma)^{(0)}$  is called aperiodic if  $G_m(\Gamma)_P^P = \{P\}$ .

Using this definition and the above proposition, we can reformulate the definition of (essentially) principality for  $G_m(\Gamma)$  given in [19] as follows:  $G_m(\Gamma)$  is principal if and only if every  $P \in G_m(\Gamma)^{(0)}$  is aperiodic.  $G_m(\Gamma)$  is essentially principal if and only if in every closed invariant set  $F$  the set of aperiodic points is dense.

**Definition 7.3.** (a) A subset  $I$  of  $\Gamma^{(0)}$  is called  $<$ -closed if  $p \in \Gamma^{(0)}$  belongs to  $I$  whenever  $p < (p_1, \dots, p_n)$  for  $p_1, \dots, p_n \in I$ .

(b) A subset  $I$  of  $\Gamma^{(0)}$  is called invariant if  $xpx^{-1}$  belongs to  $I$  for every  $p \in I$  and  $x \in \Gamma$  with  $p \prec x^{-1}x$ .

Note that an  $<$ -closed set is in particular an order ideal as  $p \prec q$  implies  $p < q$ .

**Proposition 7.4.** (a) Let  $I$  be an arbitrary subset of  $\Gamma^{(0)}$ . Then  $\text{Cl}(I) \equiv \{p : p < (p_1, \dots, p_n) \text{ for suitable } p_1, \dots, p_n \in I\}$  is the smallest  $<$ -closed subset of  $\Gamma^{(0)}$  containing  $I$ .

(b) If  $I$  is an invariant order ideal in  $\Gamma^{(0)}$  then  $\text{Cl}(I)$  is the smallest  $<$ -closed invariant subset of  $\Gamma^{(0)}$  containing  $I$ .

*Proof.* (a) It suffices to show that  $\text{Cl}(I)$  is  $<$ -closed. This follows easily from Proposition 5.6 (a).

(b) By (a), the set  $\text{Cl}(I)$  is  $<$ -closed. It remains to show invariance. Let  $p < (p_1, \dots, p_n)$  with  $p_1, \dots, p_n \in I$  and  $x \in \Gamma$  with  $p \prec x^{-1}x$  be given. This gives, by Proposition 5.6 (b),  $p = ppp < (pp_1p, \dots, pp_np) = (p_1p, \dots, p_np)$ . Applying Proposition 5.6 once more, we infer

$$(3) \quad xpx^{-1} < (xp_1px^{-1}, \dots, xp_npx^{-1}).$$

As  $I$  is an order ideal, we have  $p_jp \in I$  for  $j = 1, \dots, n$ . As  $p_jp \prec p$  we have furthermore  $p_jp \prec x^{-1}x$  and by invariance of  $I$  this gives  $xp_jpx^{-1} \in I$  for  $j = 1, \dots, n$ . Combining this with (3), we conclude (b).  $\square$

The proof of the following proposition is straightforward.

**Proposition 7.5.** (a) The set of  $<$ -closed invariant subsets of  $\Gamma^{(0)}$  with the usual inclusion as partial order is a lattice with  $I \vee J \equiv \text{Cl}(I \cup J)$  and  $I \wedge J \equiv I \cap J$ .

(b) The set of open invariant subsets of  $G_m(\Gamma)^{(0)}$  with the usual inclusion as order is a lattice with  $U \vee V \equiv U \cup V$  and  $U \wedge V \equiv U \cap V$ .

**Definition 7.6.** (a) The lattice in part (a) of the foregoing proposition will be denoted by  $\mathcal{I}(\Gamma)$ .

(b) The lattice in part (b) of the foregoing proposition will be denoted by  $\mathcal{V}(\Gamma)$ .

Next, we prove the first key result of this section.

**Lemma 7.7.** Let  $\Gamma$  satisfy (LC). For  $V$  in  $\mathcal{V}(\Gamma)$  the set  $S_i(V) \equiv \{q \in \Gamma^{(0)} : V_q \subset V\}$  belongs to  $\mathcal{I}(\Gamma)$ . For  $I \in \mathcal{I}(\Gamma)$  the set  $S_u(I) \equiv \cup_{q \in I} V_q$  belongs to  $\mathcal{V}(\Gamma)$ . The maps  $S_u : \mathcal{I}(\Gamma) \longrightarrow \mathcal{V}(\Gamma)$ ,  $I \mapsto S_u(I)$ , and  $I : \mathcal{V}(\Gamma) \longrightarrow \mathcal{I}(\Gamma)$ ,  $U \mapsto S_i(U)$ , are lattice isomorphism which are inverse to each other.

*Proof.* It is straightforward (and does not use any assumptions on  $\Gamma$ ) to show that  $S_u(I)$  belongs to  $\mathcal{V}(\Gamma)$ . Moreover, using (L),  $0 \in \Gamma$  and Proposition 6.2, it is not hard to see that  $S_i(V)$  belongs to  $\mathcal{I}(\Gamma)$ .

Let us now show that  $S_i$  and  $S_u$  are inverse to each other, i. e. that

$$(1) \ S_i(S_u(I)) = I \text{ and } (2) \ S_u(S_i(V)) = V.$$

(1). By  $S_i(S_u(I)) = \{q : V_q \subset \cup_{p \in I} V_p\}$ , we have  $S_i(S_u(I)) \supset I$ . Let conversely,  $q$  with  $V_q \subset \cup_{p \in I} V_p$  be given. By compactness of  $V_q$ , we have  $V_q \subset V_{p_1} \cup \dots \cup V_{p_n}$  for suitable  $p_1, \dots, p_n \in I$ . By Proposition 6.2, this gives  $q < (p_1, \dots, p_n)$ . As  $I$  is  $<$ -closed, we infer  $q \in I$  and the proof of (1) is finished.

(2).  $S_u(S_i(V)) = \cup_{q \in S_i(V)} V_q = \cup_{q: V_q \subset V} V_q = V$ . Here, we used in the last equation that the  $V_x$ ,  $x \in \Gamma$ , give a basis of the topology of  $G_m(\Gamma)$  by (L).

Apparently, the maps  $S_i$  and  $S_u$  respect the order. So, it remains to show that they respect  $\vee$  and  $\wedge$  as well. This will be shown next. In fact,  $S_i(U \wedge V) = S_i(U) \cap S_i(V)$  is immediate and  $S_u(I \wedge J) = S_u(I) \cap S_u(J)$  follows easily as  $p \wedge q = pq$  exists for  $p, q \in \Gamma^{(0)}$ . Thus, it remains to show  $S_u(I \vee J) = S_u(I) \vee S_u(J)$  and  $S_i(U \vee V) = S_i(U) \vee S_i(V)$ . We have

$$S_u(I \vee J) \equiv S_u(\text{Cl}(I \cup J)) = \bigcup_{q \in \text{Cl}(I \cup J)} V_q = \bigcup_{q \in I} V_q \cup \bigcup_{p \in J} V_p = S_u(I) \cup S_u(J),$$

where we used Proposition 6.2 combined with Proposition 7.4 in the previous to the last equality. Also, we have

$$\begin{aligned} S_i(U \vee V) &= \{q : V_q \subset U \cup V\} \\ (V_q \text{ compact}) &= \{q : V_q \subset V_{q_1} \cup \dots \cup V_{q_n} \cup V_{p_1} \dots \cup V_{p_k}, V_{p_j} \subset U, V_{q_i} \subset V\} \\ (\text{Prop. 6.2}) &= \text{Cl}(\{q : V_q \subset U\} \cup \{q : V_q \subset V\}) \\ &= S_i(U) \vee S_i(V). \end{aligned}$$

This finishes the proof of the lemma.  $\square$

The previous lemma characterizes the open invariant subsets of  $G_m(\Gamma)^{(0)}$  in terms of invariant  $<$ -closed subsets of  $\Gamma^{(0)}$ . An important question is whether there actually exist nontrivial invariant open subsets of  $G_m(\Gamma)$ . This question is answered in the following lemma.

**Lemma 7.8.** *Let  $\Gamma$  satisfy (LC). Then the following are equivalent.*

- (i) *There do not exist non trivial  $<$ -closed invariant subsets of  $\Gamma^{(0)}$ .*
- (ii) *For every  $p, q \in \Gamma^{(0)}$ , there exist  $x_1, \dots, x_n$  with  $x_j^{-1}x_j \prec p$ ,  $j = 1, \dots, n$ , and  $q < (x_1x_1^{-1}, \dots, x_nx_n^{-1})$ .*
- (iii)  *$G_m(\Gamma)$  is minimal, i.e. for every  $P \in G_m(\Gamma)^{(0)}$  the orbit of  $P$  is dense in  $G_m(\Gamma)$ .*

*Proof.* It is well known that an  $r$ -discrete topological groupoid  $G$  is minimal if and only if there do not exist any non trivial invariant open subsets of  $G^{(0)}$ . Thus, the equivalence of (iii) and (i) is immediate from the foregoing lemma.

It remains to show the equivalence of (i) and (ii). Obviously, (i) is equivalent to the statement that any non-empty  $<$ -closed invariant subset of  $\Gamma^{(0)}$  contains every unit. This means that for every  $p \in \Gamma^{(0)}$ ,  $p \neq 0$ , the set

$$I_p \equiv \text{Cl}(\{xrx^{-1} : r \prec p, r \prec x^{-1}x\})$$

contains every  $q \in \Gamma^{(0)}$ . This is the case if and only if for every  $q \in \Gamma^{(0)}$ , there exist  $y_1, \dots, y_n$  and  $r_1, \dots, r_n \in \Gamma^{(0)}$  with  $r_j \prec y_j^{-1}y_j, p$  and  $q < (y_1r_1y_1^{-1}, \dots, y_nr_ny_n^{-1})$ . But this is equivalent to (ii) with  $x_j \equiv y_jr_j$ ,  $j = 1, \dots, n$ , (resp.  $r_j = x_j^{-1}x_j$ , and  $y_j = x_j$ ).  $\square$

As mentioned in the introduction to this section we are interested in reductions of  $G_m(\Gamma)$  to open invariant sets. In our setting these reductions of  $G_m(\Gamma)$  can directly be described in terms of certain subsemigroups of  $\Gamma$ . This will be shown next.

It is not hard to see that, for each invariant  $I \subset \Gamma^{(0)}$  the set  $\Gamma_I \equiv \{x : xx^{-1} \in I\} = \{x : x^{-1}x \in I\}$  with multiplication and involution from  $\Gamma$  is an inverse subsemigroup of  $\Gamma$ .

**Proposition 7.9.** *Let  $\Gamma$  be an inverse semigroup with zero satisfying (L). Let  $I$  be an invariant order ideal in  $\Gamma^{(0)}$ . Then  $V(I) \equiv \cup_{q \in I} V_q$  is an invariant open subset of  $G_m(\Gamma)^{(0)}$ . The canonical embedding  $j : \Gamma_I \longrightarrow \Gamma$ ,  $x \mapsto x$  induces an isomorphism  $J : G_m(\Gamma_I) \longrightarrow G_m(\Gamma)_{V(I)}$ ,  $X \mapsto [\{j(y) : X \prec y\}]$ , topological groupoids.*

*Proof.* As in the proof of Lemma 7.7 we infer that  $V(I)$  is open and invariant. Direct calculations show that  $J : G_m(\Gamma_I) \longrightarrow G_m(\Gamma)$  and  $P : G_m(\Gamma)_{V(I)} \longrightarrow G_m(\Gamma_I)$ ,  $X \mapsto [\{x \in \Gamma_I : X \prec x\}]$  are continuous groupoid homomorphism which is inverse to each other.  $\square$

This proposition allows one to identify  $C_{\text{red}}^*(G_m(\Gamma_I))$  with  $C_{\text{red}}^*(G_m(\Gamma)_{V(I)})$  which in turn can canonically be considered as an ideal in  $C_{\text{red}}^*(G_m(\Gamma))$  by the results of [19] mentioned at the beginning of this section. Using this identification, we can state the results on the ideal structure of  $C_{\text{red}}^*(G_m(\Gamma))$ . We will denote the lattice of ideals of  $C_{\text{red}}^*(\Gamma)$  by  $\mathcal{I}(C_{\text{red}}^*(\Gamma))$ .

**Theorem 4.** *Let  $\Gamma$  satisfy (LC). Assume that  $G_m(\Gamma)$  is essentially principal. Then the following holds.*

- (a) *The map  $J : \mathcal{I}(\Gamma) \longrightarrow \mathcal{I}(C_{\text{red}}^*(\Gamma))$ ,  $J(I) \equiv C_{\text{red}}^*(G_m(\Gamma_I)) \subset C_{\text{red}}^*(G_m(\Gamma))$  is a bijection of lattices.*
- (b) *The  $C^*$ -algebra  $C_{\text{red}}^*(G_m(\Gamma))$  is simple if and only if for every  $p, q \in \Gamma^{(0)}$ , there exist  $x_1, \dots, x_n$  with  $x_j^{-1}x_j \prec p$ ,  $j = 1, \dots, n$ , and  $q < (x_1x_1^{-1}, \dots, x_nx_n^{-1})$ .*

*Proof.* (a) This follows from Lemma 7.7 and the corresponding results of chapter II, Section 4 of [19] (cf. Corollary 4.9 of [20] as well).

(b) This follows from (a) and Lemma 7.8.  $\square$

## 8. APPLICATION TO GRAPHS

In this section we present an inverse semigroup based approach to the groupoids  $G(\mathbf{g})$  associated to graphs  $\mathbf{g}$ . Combined with the results of the previous sections, this will provide semigroup based proofs for some results of [12] concerning the structure of the open invariant subsets of  $G(\mathbf{g})^{(0)}$ . Let us also mention that the ideal theory of [12] has recently extended by Paterson [17] to a non locally finite situation. He introduces the universal groupoid associated to graph inverse semigroups and then studies the graph groupoid as obtained by a reduction process.

Let  $\mathbf{g} = (E, V, r, s)$  be a directed graph [12] with set of edges  $E$  and set of vertices  $V$  and the range and source map  $r, s : E \longrightarrow V$ . We assume that  $r$  is onto and that  $s^{-1}(v)$  is not empty for each  $v \in V$ . Moreover, we assume that the graph  $\mathbf{g}$  is row finite, i.e., that  $s^{-1}(v) \subset E$  is finite for all  $v \in V$ .

A path  $\alpha$  of length  $|\alpha| = n \in \mathbb{N}$  is a sequence  $\alpha = (\alpha_1, \dots, \alpha_n)$  of edges  $\alpha_1, \dots, \alpha_n$  in  $E$  with  $s(\alpha_{j+1}) = r(\alpha_j)$ ,  $j = 1, \dots, n-1$ . For such an  $\alpha$  we set  $s(\alpha) \equiv s(\alpha_1)$  and  $r(\alpha) \equiv r(\alpha_n)$ . A path of length 0 is just a vertex and will also be called a degenerate path. For such a path  $v$ , we set  $r(v) \equiv v$  and  $s(v) \equiv v$ .

The set of all paths of finite length is denoted by  $F(\mathbf{g})$ . The set of all infinite paths  $\alpha = (\alpha_1, \dots)$  is denoted by  $P(\mathbf{g})$ . The concatenation  $\alpha\mu$  of two finite paths  $\alpha$  and  $\mu$  with  $r(\alpha) = s(\mu)$  is defined in the obvious way, i.e. by

$$\alpha\mu \equiv \begin{cases} (\alpha_1, \dots, \alpha_{|\alpha|}, \mu_1, \dots, \mu_{|\mu|}) & : |\alpha|, |\mu| > 0 \\ \alpha & : |\mu| = 0 \\ \mu & : |\alpha| = 0 \end{cases}$$

By a slight abuse of language we write  $\alpha \prec \beta$  if  $\beta = \alpha\mu$ .

We will now introduce an inverse semigroup associated to  $\mathbf{g}$ . This inverse semigroup is slightly more general than the inverse semigroup discussed e.g. in [16] in that we also allow for paths of lengths zero.

Let the set  $\Gamma \equiv \Gamma(\mathbf{g})$  be given by  $\Gamma \equiv \{(\alpha, \beta) \in F(\mathbf{g}) \times F(\mathbf{g}) : r(\alpha) = r(\beta)\} \cup \{0\}$ . Let us now define a multiplication on  $\Gamma$ : For a pair  $((\alpha, \beta), (\gamma, \delta))$  with  $\gamma = \beta\mu$  with a suitable (possibly degenerate)  $\mu$ , we define the product by

$$(\alpha, \beta)(\gamma, \delta) = (\alpha, \beta)(\beta\mu, \delta) \equiv (\alpha\mu, \delta).$$

For a pair  $((\alpha, \beta), (\gamma, \delta))$  with  $\beta = \gamma\mu$  with a suitable (possibly degenerate)  $\mu$  we define the product by

$$(\alpha, \beta)(\gamma, \delta) = (\alpha, \gamma\mu)(\gamma, \delta) \equiv (\alpha, \delta\mu).$$

In all other cases we define the product to be 0. It is then not hard to show that  $\Gamma$  is indeed an inverse semigroup, where the inverse of  $(\alpha, \beta)$  is given by  $(\alpha, \beta)^{-1} \equiv (\beta, \alpha)$ . Thus,  $\Gamma$  gives rise to a groupoid  $G_m(\Gamma)$ . Now, the function  $R : \Gamma \rightarrow [0, \infty)$  given by

$$R(\alpha, \beta) \equiv \begin{cases} 0 & : \alpha_{|\alpha|} \neq \beta_{|\beta|} \\ \sup\{j \in \mathbb{N}_0 : \alpha_{|\alpha|-i} = \beta_{|\beta|-i}, i = 0, \dots, j\} & : \text{otherwise} \end{cases}$$

can easily be seen to be a radiusfunction in the sense of Section 6. Moreover, it is possible to show that  $R$  is admissible and continuous. Thus,  $G_m(\Gamma)$  is a groupoid with a basis consisting of compact sets. We refrain from giving details, but rather show the connection between  $G_m(\Gamma)$  and the graph groupoids which were introduced in [12].

To do so, we start with a simple proposition giving an understanding of the relation  $\prec$  in the case at hand.

**Proposition 8.1.** *The relation  $(\gamma, \delta) \prec (\alpha, \beta)$  holds if and only if there exists a (possibly degenerate)  $\mu \in F(\mathbf{g})$  with  $\gamma = \alpha\mu$  and  $\delta = \beta\mu$ .*

*Proof.* This is straightforward.  $\square$

From this proposition we immediately infer the following interesting fact concerning the order structure of  $\Gamma$ .

**Proposition 8.2.** *Let  $x, y \in \Gamma$  with a common predecessor be given. Then either  $x \prec y$  or  $y \prec x$ .*

Thus, to every  $X \in \mathcal{O}(\Gamma)$ , we can find  $(\alpha, \beta) \in \Gamma$ ,  $I = [0, a] \subset \mathbb{Z}$ ,  $a \in \mathbb{N} \cup \{\infty\}$ , and edges  $e_n$ ,  $n \in I$ , s.t.  $\{(\alpha e_1 \dots e_n, \beta e_1 \dots e_n) : n \in I\}$  is a representative of  $X$ .

Putting this together, we see that minimal elements in  $\mathcal{O}(\Gamma)$  can be identified with double paths of infinite length which agree from a certain point on. But this is exactly the way the graph groupoid in [12] is constructed. Let us sketch the construction and give a precise proof of the isomorphy. Two paths  $x, y \in P(\mathbf{g})$  are called equivalent with lag  $k \in \mathbb{Z}$ , written as  $x \sim_k y$ , if there exists an  $N(x, y) \in \mathbb{N}$  s.t.  $x_i = y_{i+k}$  for all  $i \geq N(x, y)$ . Let

$$G(\mathbf{g}) \equiv \{(x, k, z) \in P(\mathbf{g}) \times \mathbb{Z} \times P(\mathbf{g}) : x \sim_k z\}.$$

Let the set of composable pairs of  $G(\mathbf{g})$  consist of all pairs  $((x, k, y_1), (y_2, l, z))$  with  $y_1 = y_2$ . For such a pair define the multiplication by

$$(x, k, y_1)(y_2, l, z) \equiv (x, k + l, z).$$

An inverse map on  $G(\mathbf{g})$  is given by letting

$$(x, k, y)^{-1} \equiv (y, -k, x).$$

Then  $G(\mathbf{g})$  together with this multiplication and inverse map is a groupoid. To make  $G(\mathbf{g})$  into a topological groupoid one introduces the sets

$$Z(\alpha, \beta) \equiv \{(x, k, y) \in G(\mathbf{g}) : \alpha \prec x, \beta \prec y, k = |\beta| - |\alpha|, x_i = y_{i+k}, i \geq |\alpha|\},$$

where  $(\alpha, \beta)$  is an arbitrary element in  $F(\mathbf{g}) \times F(\mathbf{g})$  with  $r(\alpha) = r(\beta)$ . These sets form a basis for a locally compact topology on  $G(\mathbf{g})$ . Each set  $Z(\alpha, \beta)$  is a compact and open  $G(\mathbf{g})$ -set. It is not hard to show that the system  $\mathcal{Z}(\mathbf{g})$  of all these sets is in fact an inverse semigroup.

**Theorem 5.** *The map  $j : G(\mathbf{g}) \longrightarrow G_m(\Gamma(\mathbf{g}))$ ,  $j((x, k, y)) := [\{(x_1, \dots, x_n, y_1, \dots, y_{n+k}) : n \geq N(x, y)\}]$ , is an isomorphism of topological groupoids.*

*Proof.* By definition of  $G(\mathbf{g})$  and the previous proposition, the set  $\{(x_1, \dots, x_n, y_1, \dots, y_{n+k}) : n \geq N(x, y)\}$  is directed and contains elements of arbitrary lengths. Thus,  $j(X)$  is minimal. Direct calculations show that  $j$  is a groupoid homomorphism. Now, for each  $X \in G_m(\Gamma(\mathbf{g}))$  we can find an  $(\alpha, \beta) \in \Gamma(\mathbf{g})$  with  $X \prec (\alpha, \beta)$ . As  $X$  is minimal and  $s^{-1}(v) \neq \emptyset$  for each vertex  $v$ , there exists for each  $n \in \mathbb{N}$  an path  $\mu_n$  of length  $n$  with  $X \prec (\alpha\mu_n, \beta\mu_n)$ . By the previous proposition

$$(4) \quad (\alpha\mu_{n+1}, \beta\mu_{n+1}) \prec (\alpha\mu_n, \beta\mu_n)$$

for every  $n \in \mathbb{N}$ . Then,  $\{(\alpha\mu_n, \beta\mu_n) : n \in \mathbb{N}\}$  is directed and  $X' := [\{(\alpha\mu_n, \beta\mu_n) : n \in \mathbb{N}\}]$  is minimal, as it contains paths of arbitrary lengths. As by construction  $X \prec X'$ , we infer  $X = X'$ . Moreover, by (4), we can form the “limit”  $x$  of the paths  $\alpha\mu_n$ , the “limit”  $y$  of the paths  $\beta\mu_n$  and define the map  $h : G_m(\Gamma(\mathbf{g})) \longrightarrow G(\mathbf{g})$  via

$$h(X) := (x, k, y)$$

with  $k := |\alpha| - |\beta|$ . By construction,  $h$  is inverse to  $j$ . Moreover, it is not too hard to see that  $h$  is a homomorphism of groupoids. Thus,  $j$  is an isomorphism of groupoids.

Direct calculation show that  $j(Z(\alpha, \beta)) = V_{(\alpha, \beta)}$  and we see that  $j$  and  $h$  are continuous.  $\square$

This theorem allows us to apply the theory of the preceeding sections to the study of graph groupoids. In particular, we can rephrase the ideal theory of [12] (viz the characterization of open invariant subsets of  $G(\mathbf{g})^{(0)}$ ) in terms of inverse semigroups using Section 7. This is done next.

Recall the following definitions from [12]. For vertices  $v, w \in V$  we write  $v \geq w$  if there exists a path in  $P$  from  $v$  to  $w$ . A subset  $H$  of  $V$  is called hereditary if  $v \in H$  and  $v \geq w$  implies  $w \in H$  and it is called saturated if

$$[r(e) \in H \text{ for all } e \in E \text{ with } s(e) = v] \text{ implies } v \in H.$$

The set of hereditary and saturated subsets of  $V$  is a lattice under the operation of intersection of sets and union followed by saturation. Now, we have the following lemma.

**Lemma 8.3.** *The map  $I \mapsto \{r(p) : p \in I\}$  is an isomorphism between the lattice of invariant  $\prec$ -closed ideals in  $\Gamma^{(0)}$  and the lattice of hereditary saturated subsets of  $V$ . The inverse is given by  $H \mapsto \{p \in \Gamma^{(0)} : r(p) \in H\}$ .*

*Proof.* This follows by direct arguments.  $\square$

## 9. APPLICATION TO TILINGS

As mentioned in the introduction, our study is motivated by work of Kellendonk [7, 8] introducing inverse semigroups in the context of tilings, see [9, 10] for recent work on this.

Here, we shortly discuss how the groupoid arises from the inverse semigroup in this context. This follows essentially [7] (with the slight variation that we work with directed sets rather than directed sequences). We then, apply the general theory developed above to describe the ideal structure of  $C_{\text{red}}^*(G_m(\Gamma))$  for  $\Gamma$  arising from aperiodic tilings. While this is essentially, known it serves as a good example for our theory. Moreover, it underlines the structural similarities between tilings and graphs.

A tiling in  $\mathbb{R}^d$  is a (countable) cover  $T$  of  $\mathbb{R}^d$  by compact sets which are homeomorphic to the unit ball in  $\mathbb{R}^d$  and which overlap at most at their boundaries [5]. The elements of  $T$  are called tiles. A pattern  $P$  in  $T$  is a finite subset of  $T$ . For patterns  $P$  and tilings  $T$  and  $x \in \mathbb{R}^d$ , we define  $P+x$  and  $T+x$  in the obvious way. The set of all patterns which belong to  $T+x$  for some  $x \in \mathbb{R}^d$ , will be denoted by  $P(T)$ . All patterns will be assumed to be patterns in  $P(T)$  if not stated otherwise.

A doubly pointed pattern  $(a, P, b)$  (over  $T$ ) consists of a pattern  $P \in P(T)$  together with two tiles  $a, b \in P$ . We say that  $(a, P, b)$  is contained in  $(c, Q, d)$ , written as  $(a, P, b) \subset (c, Q, d)$ , if  $a = c$ ,  $b = d$  and  $P \subset Q$ . On the set of doubly pointed patterns over  $T$  we introduce an equivalence relation by defining  $(a, P, b) \sim (c, Q, d)$  if and only if there exists an  $r \in \mathbb{R}^d$  s.t.  $c = a+r$ ,  $d = b+r$  and  $Q = P+r$ . The class of  $(a, P, b)$  will be denoted by  $\overline{(a, P, b)}$ . Obviously, the relation  $\subset$  can be extended to these classes.

Similarly, one can introduce an equivalence relation on the set of all patterns in  $P(T)$ . Denote the class of the pattern  $P$  up to translation by  $\overline{P}$  and denote the set of all classes of patterns in  $P(T)$  by  $\overline{P(T)}$ . Following [7, 16], we will assume two finite type conditions:

- (i)  $d_{\max} \equiv \sup\{\text{diam}(A) : A \in T\} < \infty$ .
- (ii) The set  $\{\overline{P} \in \overline{P(T)} : \text{diam}(\cup_{t \in P} t) \leq R\}$  is finite for every  $R$ .

Here,  $\text{diam}(A)$  denotes the diameter of  $A$ . Note that these conditions imply in particular that there only finitely many different tiles up to translation. As each tile is homeomorphic to the unit ball, this implies in particular that there is a minimal volume  $V_{\min} > 0$  among the volumes of the tiles.

Following [7], we make  $\Gamma \equiv \{\overline{(a, P, b)}; P \in P(T), a, b \in P\} \cup \{0\}$  into an inverse semigroup in the following way. A pair  $(E, F) \in \Gamma \times \Gamma$  is said to be composable if there exists a doubly pointed pattern class  $G$  and representatives  $(a, P, b)$  of  $E$ ,  $(c, Q, d)$  of  $F$  and  $(r, R, s)$  of  $G$  together with a tile  $t \subset R$  with

$$(a, P, b) \subset (r, R, t) \quad \text{and} \quad (c, Q, d) \subset (t, R, s).$$

Let  $H$  be the smallest w.r.t.  $\subset$  doubly pointed pattern class with this property. It is not hard to see that  $EF \equiv H$  is well defined. Now, we can define a multiplication on  $\Gamma$  by  $EF = H$  if  $E, F$  are composable and by  $EF = 0$  otherwise.

It can be shown that  $\Gamma$  with the above multiplication is indeed an inverse semigroup with inverse map given by  $\overline{(a, P, b)}^{-1} \equiv \overline{(b, P, a)}$ .

Moreover, the relation “ $\prec$ ” induced from the almost-groupoid agrees with the relation “ $\subset$ ” defined above [7], i.e. the following is valid:

**Proposition 9.1.** *For  $x, y \in \Gamma$  the relation  $x \prec y$  holds if and only if there exist representatives  $(a, P, b)$  of  $x$  and  $(a, Q, b)$  of  $y$  with  $P \supset Q$ .*

Of course,  $\Gamma$  gives now rise to a groupoid  $G_m(\Gamma)$ . This groupoid can easily be identified with the groupoid  $G(T)$  defined as follows [7]: Let  $\mathcal{T} = \mathcal{T}(T)$  be the set of all tilings  $S$  of  $\mathbb{R}^d$  with  $\overline{P(S)} \subset \overline{P(T)}$ . Let  $G(T)$  denote the set of all equivalence classes of doubly pointed tilings of  $\mathcal{T}$ . Here, a doubly pointed tiling and the equivalence relation are defined by just replacing the pattern  $P$  in the corresponding definitions above by a tiling  $S \in \mathcal{T}$ . The set  $G(T)$  has a groupoid structure. Two elements  $E, F$  are composable if there exists representatives  $(a, S, b)$  of  $E$  and  $(c, R, d)$  of  $F$  with  $S = R$  and  $b = c$ . In this case one defines  $EF \equiv \overline{(a, S, d)}$ . This is well defined. The topology on  $G(T)$  is generated by the sets

$$V(\overline{(a, P, b)}) = \{E \in G(T) : \overline{(a, P, b)} \subset E\}.$$

These sets are in fact compact, open  $G(T)$ -sets forming a basis of the topology. As in [8], one can then show that  $G_m(\Gamma) = G(T)$ .

$\Gamma$  admits a complete radius function  $R$ , where  $R(\overline{(a, P, b)})$  is defined by  $R(\overline{(a, P, b)}) \equiv [\text{dist}(\partial P, \{a, b\})]$ . Here,  $\partial P$  is the boundary of  $P$ .

Let us now study the structure of open invariant sets in  $G_m(\Gamma)$ . A subset  $S$  of  $\overline{P(T)}$  is called saturated if  $\overline{P} \in \overline{P(T)}$  and  $\overline{Q} \in S$  with  $\overline{Q} \subset \overline{P}$  implies  $\overline{P} \in S$ . A subset  $S$  of  $\overline{P(T)}$  is called hereditary if  $\overline{P}$  belongs to  $S$ , whenever there exist  $\overline{P}_1, \dots, \overline{P}_n$  in  $S$  satisfying the following condition:

- For every pattern  $Q$  with  $R(Q)$  large enough and  $Q \supset P$ , there exists  $j \in \{1, \dots, n\}$  with  $Q \supset P_j$ .

Then, we can easily infer the following lemma.

**Lemma 9.2.** *The map  $I \mapsto \{\overline{P} : \overline{(a, P, b)} \in I\}$  is an isomorphism of the lattice of invariant  $\prec$ -closed subsets of  $\Gamma^{(0)}$  and the lattice of saturated hereditary subsets of  $\overline{P(T)}$ .*

It remains to study principality of  $G_m(\Gamma)$ . Here, we have a very simple and well-known condition. Recall, that a tiling  $S$  is called periodic if there exists an  $x \in \mathbb{R}^d$  with  $S + x = S$ . Now,  $\mathcal{T}$  is called aperiodic if it does not contain a periodic tiling.

**Lemma 9.3.**  *$G_m(\Gamma)$  is principal if and only if  $\mathcal{T}$  is aperiodic.*

*Proof.*  $G_m(\Gamma)$  is principal, if every  $P$  in  $G_m(\Gamma)^{(0)}$  is aperiodic in the sense of Section 7. But this can easily be seen to be equivalent to  $\mathcal{T}$  being aperiodic in the sense given above.  $\square$

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